PART I

FOUNDATIONS

CHAPTER 1

THE FIRST GEOMETERS

The exact origin of the ancient civilization of **Egypt** is unknown, but it definitely dates to sometime before the year 4000 B.C. There were two kingdoms along the Nile, an upper kingdom and a lower. It is said that somewhere between the years 3500 to 3000 B.C. the legendary Menes unified the two kingdoms and so formed the First Dynasty. He is further claimed to have founded the city of Memphis and dedicated a temple there to the creator-god called Ptah. For the next 500 to 1000 years Egypt grew in strength as a state and a society. The Third Dynasty, which lasted from 2700 to 2625 B.C, saw the reign of King Djoser. With the assistance of his chief architect and physician, Imhotep, they directed the construction of the first great stone building, the Step Pyramid at Saqqara. It sat in the middle of a great mortuary complex and was intended to see Djoser safely into the afterlife. During the Fourth Dynasty, approximately a half century later, Khufu (Cheops) reigned. Little is known about this pharaoh. The few texts that speak of Khufu range from portraying him as a amicable monarch interested in magic to a cruel slave master driven to build a magnificent monument to himself. This monument was the Great Pyramid. It currently stands the tallest of a group of three pyramids on the Giza plateau and is the only surviving member of the Seven Wonders of the Ancient World. It originally stood 481 feet tall but has lost 30 feet off its height. Its square base has sides that measure about 755 feet each with the difference between the longest and shortest sides being only 8 inches. Scholars estimate that it was built using 2.3 million stone blocks. The average weight of each is around 2.5 tons. In July 1798, Napoleon calculated that the stones from the three pyramids at Giza could be used to build a wall around France 1 foot wide and 12 feet high (Grimal 1992, 63–67, 389; Boardman 1982, 14, 145; Clayton 1995, 37, 46).

According to the ancient Greek historian Herodotus, there was a great king of Egypt whose name was Sesostris. It was claimed that he conquered vast portions of Asia and Europe and erected pillars on which were inscribed his name and country. After he had conquered the world, Sesostris returned home and focused on domestic issues. One of his major achievements was supposedly an extensive canal system that brought water from the Nile to his people. It is unlikely that this Sesostris ever existed. There were pharaohs who shared his name. Sesostris I (Senusret I) reigned at the beginning of the 12th Dynasty (*c.* 1990 B.C.), and he was succeed by his son, Amenemhat II, who was followed by his son, Sesostris II, and then his grandson, Sesostris III. However, neither of these can be the Sesostris of Herodotus. Instead, the pharaoh described by Herodotus was probably a legend, based on the reigns of Rameses II and Sety I, that later Egyptians created at a time when the power of Egyptian civilization began to wane (Herodotus 1987, II; Shaw 2000, 160–166, 295–302).

To the northeast of Egypt between the Tigris and Euphrates rivers was the land of Mesopotamia. This was the location of the fabled cities of Babylon, Ur, Susa, and Kish. About 4000 B.C. the **Sumerians** migrated to Mesopotamia and settled in its southern regions. Their land became known as Sumer and its capital was Ur. Around 2200 B.C. they lost their independence to the **Akkadians**, who were Semites from northern Mesopotamia. At the time the Akkadians were ruled by Sargon. His political life began in the court of King Ur-Zababa of Kish. Sargon rose to power and eventually gained control of Sumer. Many cuneiform tablets laud Sargon as a great ruler and warrior who supposedly conquered Asia Minor and the Mediterranean. Although the Sumerians and the Akkadians had there own identities, it is common to refer to them under the single term **Babylonian** (Leick 2003, 6, 102, 113; Crawford 2004, 29, 33).

Probably the best known ruler of Mesopotamia was Hammurabi. He reigned from 1792 to 1750 B.C. as king of Babylon during the First Dynasty. Beginning with a small region around the city, Hammurabi gradually extended his domain. This was accomplished by a combination of military conquests and temporary diplomatic alliances that were ended in betrayal by the king. By 1755 B.C., after 30 years of strengthening his position, Hammurabi had gained control of all of Mesopotamia, and Babylon would become the leading power in western Asia. As he accomplished this, Hammurabi established a strong central government, invested in irrigation, and strengthened the walls of the city of Babylon. In order to establish justice throughout his realm, a sequence of laws was published in all of his major cities. Known as the Code of Hammurabi, it contained 282 separate laws, covering everything from serious crimes such as murder to everyday business transactions. Several copies survive, including one found in 1902 at Susa on an 8-foot-tall stone monument of polished black diorite. The death of Hammurabi saw the beginning of the decline of the First Dynasty, which fell when Mursili, king of the Hittites, attacked Babylon by surprise in 1595 B.C. or 1499 B.C. (Leick 2003, 47–48, 57–58).

Although there was some contact between the Egyptian and Mesopotamian civilizations, on the other side of the continent in **China** there developed a society independent of Western contacts. The Chinese civilization began as a collection of small city-states along the banks of the Yellow River. Near the end of the twelfth century B.C., an ancient tribe began to grow in strength. It later became known as the Zhou dynasty and was led by King Wen, a well-respected leader who had ambitions to conquer the Shang dynasty, which was then dominant. King Wen died before he could finish his plan, but his son, Prince Wu, would later join forces with neighboring tribes to completely defeat the Shang (Shaughnessy 1999, 307–309).

The now King Wu established the Zhou dynasty with Gaojing as its capital. This dynasty lasted from 1122 to 256 B.C. This long time period is divided into two periods, the first being the Western Zhou (1122–771 B.C). The Western Zhou society was feudal in the sense that the king appointed rulers as his representatives over local jurisdictions. However, the relationship between subject and king was modeled on that of kinship as opposed to the impersonal, contractual relationship that arose in Europe. This system allowed the land over which the dynasty exercised control to expand, but it expanded too quickly and led to a collapse. This resulted in the Zhou capital's moving to Luoyi. This is the beginning of the Eastern Zhou period. This second period is divided further into the Spring and Autumn Period (771–481 B.C) and the Warring States Period (402–221 B.C). Although both periods saw many wars where states vied for power, significant cultural progress was made. For example, iron was introduced, new farming and military techniques were developed which included the use of horses, and writing was further developed (Roberts 1999, 9–12).

During the Warring States Period the Qin state, which had existed since the 8th century B.C., developed into a power that could rival the Zhou. The Qin were led by Qin Shi Huangdi who wished to unify all of China under his leadership. At the time there were six other states that were in a seemingly constant state of war. They were all defeated by Qin Shi Huangdi, ushering in the Qin dynasty and bringing order and unification to China. The dynasty only lasted from 221 B.C. to 206 B.C., but it was a significant period in Chinese history during which Chinese society was restructured so that the peasants now owned the land but were taxed by the government and a new code of laws was established which applied to all equally. In addition, Qin Shi Huangdi standardized weights, measures, and the font used in writing, and he led the construction of many civil projects, including a network of walls which led to construction of the Great Wall. However, to control knowledge the emperor ordered the destruction of many books and ordered the execution of any scholar who dissented. By the end of his reign Qin Shi Huangdi had became a tyrant. At the time of his death in 210 B.C., the people had grown dissatisfied living under the harsh conditions imposed by the emperor's construction projects. Millions had been drafted to build the projects, and this led to a rebelion. Qin Shi Huangdi's successor was killed, after which the Qin quickly surrendered. The year was 206 B.C., and civil war soon followed. The winner was a leader of the peasants named Liu Bang. His victory led to the establishment in 205 B.C. of the Han dynasty which lasted for 426 years. During this time a strong central government was established, yet many of the harsh policies of the Qin were removed and taxes reduced. This was also a time when the teachings of Confucius were made the state religion, the arts and literature flourished, and Chinese philosophy was at its peak (Roberts 1999, 19–29).

1.1 EGYPT

It is believed by most historians of mathematics that the Egyptians began studying geometry due to a need for a reliable method of measuring areas of land. This view is supported by the testimony of Herodotus.

> The priests also say that it was this king [Sesostris] who divided the land among all the Egyptians, giving to each man as an allotment a square, equal in size; for the king derived his revenues, as he appointed the payment therefore of a yearly tax. If the river should carry off a portion of the allotment, the man would come to the king himself and signify what had happened, whereupon the king sent men to inspect and remeasure by how much the allotment had grown less, so that for the future it should pay proportionally less of the assigned tax. I think it was from this that geometry was discovered and came to Greece. (Herodotus 1987, 2.109, 175)

The ancient Egyptians used lengths of rope with knots at regular intervals to measure land. By stretching the ropes along the needed dimensions and applying the appropriate formula the area would be found. From this practice the subject of geometry received its name. It is a combination of two Greek words: *geo*, meaning "earth," and *metron*, meaning "measure" (Burton 1985, 56–57).

What we know of Egyptian geometry can be traced to a surprising source. In 1798, Napoleon invaded Egypt in an attempt to protect French trade interests in the region and to weaken British ties to India. Napoleon brought a group of scholars along with his army. This was evidently a propaganda ploy to cast a favorable light on his intentions. Whatever his true motivation, his decision led to the discovery of the **Rosetta Stone** in 1799. This is a stone slab on which is carved a message in Greek, in Demotic, and in the Egyptian writing called **hieroglyphics**, a picture-based system that was used in formal settings. The meaning behind the Egyptian hieroglyphics had long eluded scholars. Finding the Rosetta Stone unlocked their meaning since they were now able to compare the hieroglyphics with ancient Greek (Boyer and Merzbach 1991, 10–11; Burton 1985, 36).

In 1858, Scotland native A. Henry Rhind made another discovery. In the tombs of Thebes in Luxor he found a scroll and purchased it for his collection. Scholars estimate that it was written about 1650 B.C., placing its creation during the reign of King Apophis I of the 17th Dynasty. The author of the scroll, Ahmes, claimed that his work was a copy of a document from the Third Dynasty, possibly originating from Imhotep. The scroll material was **papyrus**, sheets made by pressing the pith of the papyrus plant and then slicing off the pieces that were needed. Since it is organic, papyrus is not a good material for preservation purposes. Although the climate in Egypt is dry, which helped to preserve the scrolls, most finds have deteriorated to some degree. This was the case with Rhind's discovery. The scroll was 1 foot wide and 18 feet long but missing its middle portions. The content of the scroll that remained was seen to be written in the ancient Egyptian writing called **hieratic**. Beginning as a simple modification of hieroglyphics, hieratic later became a language in which each syllable is represented as a symbol and each symbol represents a particular idea. Because of this connection to hieroglyphics, the Rosetta Stone made possible the translation of the scroll that we know as the **Rhind Mathematical Papyrus** (Kline 1972, 15–16).

About four years after Rhind found his papyrus, the American Edwin Smith was studying in Egypt when he purchased what he believed to be a scroll on ancient Egyptian medicine. However, it was a fraud. Someone had taken pieces from a papyrus and created what looked like a scroll, but only the outside layers were ancient Egyptian. The inside was fake. Smith was an expert on Egypt, and it seems that it would be unlikely for someone in his position to be fooled by a forgery. We must remember, however, that the condition of any found papyri will be very fragile. It takes time and care to open an ancient scroll. Fortunately for posterity, Smith kept his find, and it went with the rest of his collection to the New York Historical Society after his death. In 1922 experts at the museum determined that the fraudulent scroll was assembled using the lost pieces of Rhind's papyrus! The pieces were shipped to the British Museum, where experts were able to fit together most of the missing section and complete its translation (Burton 1985, 36).

Other papyri discovered are the Golenischev Papyrus, usually called the **Moscow Papyrus** because it is now a part of the collection of the Museum of Fine Arts in Moscow, and a second purchase of Rhind's, which we know as the **Egyptian Mathematical Leather Scroll**. All of these finds have greatly aided our understanding of ancient Egyptian mathematics. The ancient Egyptian's wrote their mathematics in the form of questions followed by answers. The goal was to calculate values associated with government and commerce, specifically the computation of areas and volumes. They used specific numbers to solve problems, and they did not generalize their results. Any formulas that they did use were not deduced from first principles but were discovered empirically by trial and error. Because of this, some of their formulas were not accurate but only approximations, which, considering their needs, often sufficed. The greatest mystery regarding the mathematics of the ancient Egyptians is that for about two millennia they made very little progress beyond their initial work (Burton 1985, 36, 52, 57, 64).

Approximations

The mathematical scrolls show that the Egyptians did not distinguish between algebra and geometry. They viewed mathematics simply as a tool by which to obtain needed values. They had formulas for basic areas and volumes. This included the volumes of a cube and a cylinder. There is some debate in academic circles as to whether the Egyptians ever gave any justifications for any of the steps in their solutions. None have been discovered, but it is clear that the Egyptians did not consider geometry to be a logical system (Kline 1972, 19–20).

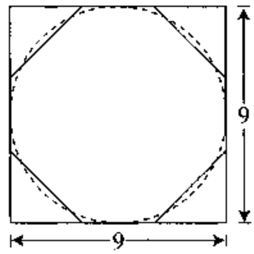


Figure 1.1 Using a 9×9 square to estimate the area of a circle.

The Rhind Mathematical Papyrus provides an example of an ancient Egyptian formula that did satisfactorily approximate its intended value. Take a square with sides of length 9. Trisect each of the sides, and then by joining consecutive points at the corners, create a regular octagon. This is illustrated in Figure 1.1, but we must note that the diagram found in the papyrus was a simple sketch of this diagram without the circle. The scribe of the papyrus considered the octagon a good estimate for the area of the circle. To find the octagon's area, we simply remove the four equilateral triangles from the square. Therefore, its area is

$$9^2 - 4 \cdot \frac{9}{2} = 63.$$

It is conjectured that this provided the ancient Egyptians a reason to believe that for a circle of diameter d,

$$A \approx \left(\frac{8d}{9}\right)^2. \tag{1.1}$$

The Egyptians could have relied on the octagonal estimate in support of Equation 1.1 because it yields an area of 64 square units when $d = 9$, which is close to the Rhind approximation. Furthermore, notice what Equation 1.1 implies about the Egyptian value of π. Since $A = \pi d^2/4$, we substitute Equation 1.1 to find

$$\pi \approx 3\frac{13}{81} = 3.16049\ldots. \tag{1.2}$$

This is very close to the common approximation of $3\frac{1}{7}$ (Burton 1985, 58).

An example of an equation from ancient Egypt that did not provide very good estimates is one that was found at the temple of Horus at Edfu. The priests of the temple received gifts in the form of patches of land. Each plot was in the form of a quadrilateral but was not necessarily rectangular. If we let a, b, c, and d represent the lengths of consecutive sides of the quadrilateral, their formula is

$$A = \frac{1}{4}(a + c)(b + d).$$

Notice that this amounts to multiplying the average of opposite sides together. It is not a bad estimate for regions that are nearly rectangular, but for most quadrilaterals it is an example of an instance when the Egyptians did try to generalize a result but did it poorly (Burton 1985, 57).

Pyramids

Herodotus recounts an encounter that he had with one of the priests (Burton 1985, 62–63). He claims that he was told that the Great Pyramid was designed so that the area of a square with side equal to the height of the pyramid is equal to the area of one of the faces of the pyramid. Using Figure 1.2, this is equivalent to writing

$$h^2 = \frac{1}{2}(2ba) = ab,$$

where $2b$ is the length of one of the sides of its square base, a is the altitude of one of the faces, and h is the height of the pyramid. From this we can calculate the ratio b/a by first noting that since

$$h^2 + b^2 = a^2,$$

we have

$$a^2 - b^2 = ab.$$

Dividing through by a^2 and a little rearranging yields

$$\left(\frac{b}{a}\right)^2 + \frac{b}{a} - 1 = 0.$$

This is a quadratic equation with b/a serving as the unknown. Thus, since both a and b are positive,

$$\frac{b}{a} = \frac{\sqrt{5} - 1}{2}. \tag{1.3}$$

Geometrically, a **pyramid** is a solid consisting of a collection of polygons joined at their edges. Recall that a **polygon** is a closed plane figure consisting of line segments called **sides**. An endpoint of a side is called a **vertex**. A polygon with n sides is often called an **n-gon**. If all the sides of the polygon are of the same magnitude and if the same can be said of the interior angles, the polygon is **regular**. In a pyramid one

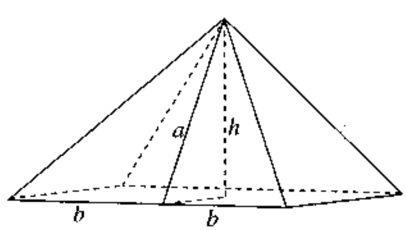


Figure 1.2 The Great Pyramid.

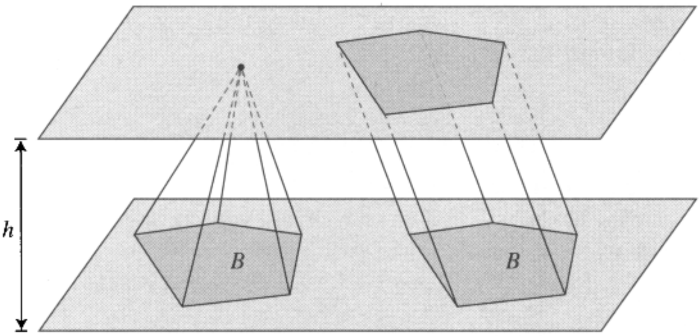


Figure 1.3 A pyramid and a prism of height h.

polygon serves as the **base** and the others are **faces**. The faces all share a common vertex called the **apex** of the pyramid. The distance between that apex and the plane containing the base is the **height** of the pyramid (Figure 1.3).

Similar in definition to a pyramid is the prism. A **prism** is a solid consisting of a collection of polygons joined at their edges. Two of the polygons are the **bases** of the prism. They are identical in shape and size (in other words, they are **congruent**) and lie in parallel planes. The other polygons are called **faces** and are formed by joining corresponding vertices of the bases (Figure 1.3). The **height** of a prism is the distance between the two parallel planes. In general, if B is the area of the base of a prism and h is its height,

$$\text{area(prism)} = Bh,$$

and the volume of a pyramid is one third the volume of a prism with the same base and height of the pyramid. This means that

$$\text{area(pyramid)} = \frac{Bh}{3}.$$

The formula for the volume of a **square pyramid** (one with a square base) quickly follows from this.

The Moscow Papyrus contains what is one of the most significant mathematical discoveries of the ancient Egyptians. The translation of Gillings (1972, 188) reads as follows:

> Method of calculating a truncated pyramid.
> If it is said to thee, a truncated pyramid of 6 [cubits] in height,
> Of 4 [cubits] of the base, by 2 on the top,
> Reckon thou with this 4, squaring. Result 16.
> Double thou this 4. Result 8.
> Reckon thou with this 2, squaring. Result 4.
> Add together this 16, with this 8, and with this 4. Result 28.
> Calculate thou [one third] of 6. Result 2.
> Calculate thou with 28 twice. Result 56.
> Lo! It is 56! Thou has found rightly.

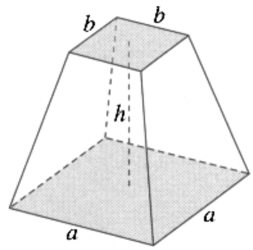


Figure 1.4 A frustum or truncated pyramid.

The Egyptians found an accurate formula for the area of a truncated square pyramid. Such a volume is called a **frustum** and is formed by taking the volume of a solid, such as a pyramid or a cone, that lies between two parallel planes. If a is the length of a side of the base of the truncated pyramid and b the length of the square at its top (Figure 1.4), then

$$V = \frac{h}{3}(a^2 + ab + b^2). \tag{1.4}$$

How Equation 1.4 was obtained is only conjecture, but a reasonable guess is that the ancient Egyptians checked such cases as when b is half of a or a third of a, and from these results concluded a formula that they could then check further in other cases (Gillings 1972, 190–191; Burton 1985, 59–60).

We can prove Equation 1.4, however, by setting up a coordinate system on a cross-section of the frustum with the origin at O (Figure 1.5). The slope of $\overline{BA}$ is $2h/(b-a)$, so the equation for the line through B and A is

$$y = \frac{2h}{b-a}x + \frac{ah}{a-b}.$$

Therefore, the height of the pyramid without the top removed is $ah/(a-b)$. The formula for the frustum is simply the volume of the large pyramid minus the volume

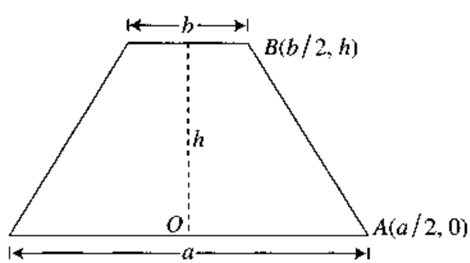


Figure 1.5 Deriving the formula for the volume of the truncated pyramid.

of the small pyramid at the tip. We calculate as follows:

$$\begin{aligned}
V &= \frac{1}{3}\left(\frac{ha}{a-b}\right)a^2 - \frac{1}{3}\left(\frac{ha}{a-b} - h\right)b^2 \\
&= \frac{h}{3}\left(\frac{a^3}{a-b} - \frac{ab^2}{a-b} + b^2\right) \\
&= \frac{h}{3(a-b)}(a^3 - ab^2 + b^2[a-b]) \\
&= \frac{h}{3(a-b)}(a^3 - b^3) \\
&= \frac{h}{3}(a^2 + ab + b^2).
\end{aligned}$$

Exercises

1. The Egyptians estimated the area of a quadrilateral that had consecutive sides of length a, b, c, and d with the formula $A = \frac{1}{4}(a+c)(b+d)$.
   (a) Show that if the quadrilateral is a rectangle, its area equals A.
   (b) Explain why A is equal to the product of the average of opposite sides of the quadrilateral.
   (c) Find the dimensions of a quadrilateral where the absolute value of the error between its true area and the estimate given by A is 25 percent.

2. Confirm that the ancient Egyptian approximation for π is as given in Equation 1.2.

3. Assuming $\pi = 3$, the ancients inaccurately used the formula

$$V = \frac{h}{12}\left(\frac{3}{2}[D+d]\right)^2$$

for the volume of a truncated cone where D is the diameter of the base, d is the diameter of the top, and h is the height. Derive the correct formula.

4. The **lateral surface area** of a solid is the total area of the sides of the solid excluding any bases. Find the lateral surface area of a truncated square pyramid of height h and bases with sides of lengths a and b.

5. From the Rhind Mathematical Papyrus we know that the Egyptians were able to calculate the slopes of the sides of a pyramid. This slope is called a **seked**. Its value is equal to the horizontal distance for every unit rise (Gillings 1972, 185–187).
   (a) Find the seked of a square pyramid of height 429 feet and with a base that is 618 feet long.
   (b) If the seked of a square pyramid is 5 feet per unit and its base is 400 feet long, what is the height of the pyramid?
   (c) Find an equation that expresses the seked in terms of the base and height of the pyramid.

6. Since the volume of a pyramid is equal to one third of the volume of a prism with base and height equal to that of the pyramid, it would have been simple for the ancient Egyptians to determine an empirical formula for a pyramid.

   (a) Describe an experiment that the Egyptians could reasonably have used to determine the formula for the volume of a pyramid.

   (b) Gillings (1972, 190) gives two conjectures concerning a mathematical approach to determining the formula. The first involves taking a right pyramid with a square base. Divide it into four congruent oblique pyramids by passing two planes through the vertex of the pyramid such that the planes are perpendicular to each other and the pyramid's base and divide the base into four equal squares. Show that the resulting pyramids can be arranged into a square from which the formula can be derived.

   (c) The second method is to start with a cube. Draw six pyramids such that each has as its base a side of the cube and each has a height that is equal to half the length of a side of the cube. It is best to draw these pyramids so that they share a common vertex on the interior of the cube. Each of these is called a **Juel's pyramid.** Use this construction to confirm the formula for the volume of a pyramid.

7. A **regular** pyramid has a regular base.

   (a) Find the lateral surface area of a regular hexagonal pyramid with height 10 and base with sides of length 4.

   (b) Find the pyramid's total area and volume.

8. Use calculus to confirm the formula for the volume of a square pyramid.

1.2 BABYLON

As the Rosetta Stone aided in the translation of the hieroglyphics of ancient Egypt, the 1870 discovery at the Behistun Cliff in Iran did the same for the Babylonian language. An inscription in the side of the cliff announced the ascension to power of Darius. It was written in the Persian, Elamitic, and Babylonian languages. The symbols were all various forms of **cuneiform**. Typically, cuneiform was written on clay tablets when they were still soft using a stylus with a triangular tip to impress wedges into the clay. This is how cuneiform received its name, for this word is from the Latin for "wedge." The clay was then dried. As opposed to papyrus, clay survives well over time, so there were many museums with large collections of tablets with cuneiform writing on them. However, they were not easily deciphered. In particular, there were large collections at the British Museum, the Louvre, Yale, Columbia, and the University of Pennsylvania, with many needing translation. Because of the Behistun Cliff and the later work of Otto Neugebauer and Paul Thureau-Dangin, the tablets were discovered to contain ancient Babylonian mathematics (Boyer and Merzbach 1991, 9–10; Kline 1972, 5).

Approximately two thirds of the clay tablets dated from 1800 to 1600 B.C, and they originated from the Sumerian and Akkadian civilizations. But for a few exceptions, the Babylonian mathematics proved to be significantly ahead of the Egyptians. Both civilizations used empirical methods to find their results, but the Babylonians were more theoretical and able to solve algebraic equations. They used symbols for unknown quantities and their equations would include one or more variables. They even knew a form of the Quadratic Formula. Despite this, their solutions were on a case-by-case basis. They had no concept of letting coefficients be represented as unknown parameters. More importantly, they had no concept of a proof. The closest they came would be a step-by-step explanation of how to solve an equation, yet this would be given without any justification (Burton 1985, 67, 69; Kline 1972, 13–14).

Pythagorean Triples

Although geometry as a subject on its own right played little role in Babylonian mathematics, there is one particular clay tablet that is of interest. Deciphered by Neugebauer and Abraham Sachs in 1945, it gave clear evidence that the Babylonians around the years 1900 to 1600 B.C. knew the Pythagorean Theorem. The tablet is called Plimpton 322 since it is housed in the G. A. Plimpton collection at Columbia University. The tablet contains a table displaying four columns of numbers, three of which are translated into decimal numbers as follows:

x	z	
119	169	1
3367	*4825	2
4601	6649	3
12709	18541	4
65	97	5
319	481	6
2291	3541	7
799	1249	8
*481	769	9
4961	8161	10
45	75	11
1679	2929	12
*161	289	13
1771	3229	14
56	*106	15

If we check some of the numbers, we find that the difference between the the square of a value from the z column and the square from the x column yields another perfect square: for instance,

$$169^2 - 119^2 = 120^2.$$

Four of the lines of the table did not follow this pattern (indicated by a *). These discrepancies have been shown to be due to errors on the part of the scribe. The right-most column served simply to enumerate the lines of the table (Burton 1985, 77–78; Kline 1972, 4–5).

The Babylonians wrote the numbers in the table using the **sexagesimal** system (Exercises 3 to 5). This type of numeration originated with the Sumerians and dates to at least 2000 B.C. Numbers written in sexagesimal are in base 60. This means that the Sumerians had symbols for the numbers 1 through 59. When 60 was reached, they would begin again at 1. They originally had no notation for zero. For example, since 119 = 60 + 59, they would write it in base 60 as

$$119 = 1, 59.$$

The number 120 was represented as 2. Because of the lack of a zero, their numbers were ambiguous and required some context for clarification. However, around the year 350 B.C. the Babylonians did introduce a symbol that would serve as a placeholder for missing digits. Their sexagesimal system was very efficient and remains with us today when we measure both time and degrees in minutes and seconds (Boyer and Merzbach 1991, 23, 27, 37).

Returning to the table we note that there is no extant explanation as to how the Babylonians obtained their result, but Burton (1985, 77–82) details a conjecture on their method. First we must note that the numbers in the table are too large for simple trial and error. If the Babylonians were guessing, we would expect to find simpler examples in addition to those already in the table. Instead, we can make sense of the numbers if we examine the fourth column of the table. The column is incomplete due to the condition of the tablet, but it can be seen that it enumerates the values of z^2/x^2. This suggests that the Babylonians took

$$x^2 + y^2 = z^2$$

and transformed it into

$$\left(\frac{z}{x}\right)^2 - \left(\frac{y}{x}\right)^2 = 1.$$

Letting $a = z/x$ and $b = y/x$, we have

$$a^2 - b^2 = 1.$$

To generate the table we simply construct triangles with sides of integral length 1, a, and b such that $a^2 - b^2 = 1$. However, we know that

$$a^2 - b^2 = (a + b)(a - b). \tag{1.5}$$

Hence, $a + b$ and $a - b$ are reciprocals, so we may write

$$a + b = \frac{m}{n} \text{ and } a - b = \frac{n}{m},$$

where m and n are positive integers. Adding the two equations, we find that

$$a = \frac{1}{2}\left(\frac{m}{n} + \frac{n}{m}\right),$$

and

$$b = \frac{1}{2}\left(\frac{m}{n} - \frac{n}{m}\right).$$

These equations simplify to

$$a = \frac{m^2 + n^2}{2mn}$$

and

$$b = \frac{m^2 - n^2}{2mn}.$$

Letting $x = 2mn$ and remembering that $y = bx$ and $z = ax$, we obtain

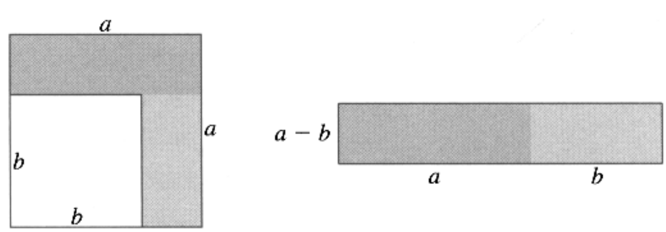
$$x = 2mn,$$
$$y = m^2 - n^2,$$
$$z = m^2 + n^2,$$

for positive integers m and n. These are well-known equations for generating **Pythagorean triples**, sets of integers that satisfy the Pythagorean Theorem.

Areas

It is unlikely that we will ever know with reasonable certainty whether the given derivation for the numbers in the table is what the Babylonians had in mind. However, it is quite certain that they knew the algebraic identity given in Equation 1.5, and how they discovered it is probably a geometric question. The expression $a^2 - b^2$ refers to the area that remains when a square with sides of length b is subtracted from one with sides of length a. The shaded regions in Figure 1.6.a represents this difference. Notice that the top shaded rectangle has dimensions a by $a - b$, and the one on the right has dimensions b by $a - b$. When we place the two rectangles side by side, forming a longer rectangle as in Figure 1.6.b, the resulting rectangle has dimensions $a + b$ by $a - b$. This confirms the identity (Burton 1985, 80).

In the 1930s a group of archeologists discovered a cache of mathematical tablets at Susa. The tablets contained one of the relatively few examples of Babylonian geometry. Not only did they include some of the oldest applications of the Pythagorean Theorem (Burton 1985, 82), but they also contained tables of ratios involving various magnitudes related to regular polygons, including the ratios between the area of



(a) $a^2 - b^2$ (b) $(a + b)(a - b)$

Figure 1.6 $a^2 - b^2 = (a + b)(a - b)$.

certain regular polygons and the square on one of its sides. The Babylonian ratios for the **pentagon** (5-gon), **hexagon** (6-gon), and **heptagon** (7-gon) are summarized in the next table (Boyer and Merzbach 1991, 37–38):

sides	area : square of side
5	1 2/3
6	2 5/8
7	3 41/60

Let us check the accuracy of their work for the pentagon. Figure 1.7 represents a portion of a pentagon. There is a point in the interior of every regular polygon called its **center** that has the property that it is equidistant from each of the endpoints of the sides. Let B be the center of the pentagon in question. Let M be the midpoint of side $\overline{AC}$. Point M has the property that $\overline{BM}$ is perpendicular to $\overline{AC}$. Every segment that joins the center to the midpoint of a side is called an **apothem** of the regular polygon. To find the area of the pentagon, we first find the area of $\triangle ABC$. The apothem a of the pentagon is the height of the triangle. The measure of the **central angle** in radians is $2\pi/5$, so $m\sphericalangle MBC = \pi/5$. Hence, letting $r = AB$, the **radius** of the polygon, we have

$$a = r\cos\frac{\pi}{5}$$

and

$$AC = 2r\sin\frac{\pi}{5}.$$

We may then conclude that

$$\text{area}(\triangle ABC) = r^2\cos\frac{\pi}{5}\sin\frac{\pi}{5} = \frac{r^2\sin 2\pi/5}{2}.$$

Denote the area of a regular n-gon with radius r by A_n. Let s_n be the length of its side. Then $s_5 = AC$ and

$$A_5 = \frac{5r^2\sin 2\pi/5}{2}.$$

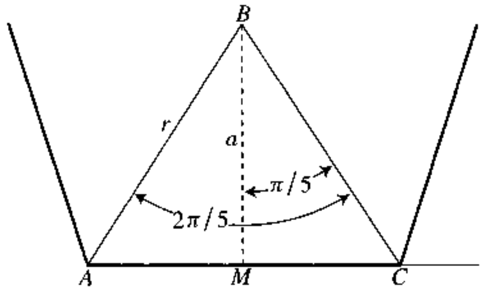


Figure 1.7 Finding the area of a regular pentagon.

This generalizes to

$$A_n = \frac{nr^2 \sin 2\pi/n}{2} \tag{1.6}$$

and

$$s_n = 2r \sin \frac{\pi}{n}. \tag{1.7}$$

Using these equations, we see that the ratio of the area of a pentagon to the square on its side is

$$\frac{A_5}{s_5^2} = \frac{5}{4 \tan \pi/5} \approx 1.72048.$$

The difference between our calculation and the value from the table is 0.05381. This amounts to a 3.12% error for the Babylonian calculation.

The clay tablets found at Susa also yielded ratios between the perimeter of a regular polygon and the circle circumscribed about it. As we did with the Egyptians, this should allow us to find the Babylonian value for π. Let us take the regular hexagon as an example. The ratio of the perimeter of a regular hexagon to the circumference of its circumscribed circle was calculated by the ancient Babylonians to be 24/25 (Boyer and Merzbach 1991, 38). Let P_n represent the perimeter of a regular polygon. By Equation 1.7,

$$P_n = 2nr \sin \frac{\pi}{n}.$$

Therefore,

$$\frac{P_6}{2\pi r} = \frac{6 \sin \pi/6}{\pi}.$$

Setting this equal to the ancient estimate, we find that

$$\pi \approx \frac{25 \cdot 6 \sin \pi/6}{24} = \frac{75}{24} = 3\frac{1}{8}.$$

The is an improvement over the early Babylonian (1800–1600 B.C) assumption of $\pi \approx 3$ and compares very favorably to the Egyptian value calculated in Equation 1.2.

Exercises

1. Show that each of the following sets of numbers are Pythagorean triples. Assume that a and b are integers.
   (a) $2ab, a^2 - b^2, a^2 + b^2$
   (b) $a, (a^2 + 1)/2, (a^2 - 1)/2$ (provided that a is odd)

2. Burton (1985, 84–85) cites a number of problems found on Babylonian tablets.
   (a) The circumference of a circle is 60 and the length of a perpendicular from the center of a chord to the circumference is 2. Find the length of the chord assuming that $\pi = 3\frac{1}{8}$.
   (b) One leg of a right triangle has length 50. Parallel to the other leg at a distance of 20, a segment is drawn that cuts off a right trapezoid of area 320. Find the lengths of the bases of the trapezoid.

(c) Given a right triangle with base of length 30 and a segment parallel to the base of length c that divides the triangle into a right triangle of height a and a right trapezoid of height b. The area of the right trapezoid is 420 more than the area of the small right triangle. Given that this information yields the equations $a - b = 20$ and

$$\frac{b(c+30)}{2} = \frac{ac+840}{2},$$

use similar triangles to find the values of a, b, and c.

3. We will use the numbers 0, 1, 2, ..., 59 separated by commas to represent integers written in sexagesimal. The number farthest right will be considered in the units (60^0) position, the next one to the left is in the 60^1 position, and so on. For example,

$$14, 43, 3 = 14 \cdot 60^2 + 43 \cdot 60^1 + 3.$$

Use this notation to convert the decimal (base 10) numbers 60, 56,861, 4,968,011, and 315,839,601 to sexagesimal.

4. Since sexagesimal is a positional numeral system, it is easy to calculate with it. (Think of the problems that arise even with simple arithmetic in the Roman numeral system.) All of the operations behave as with the decimal system, but we have to remember that it is base 60. When we "carry" or "borrow," we do it with 60s and not 10s. For example, $45 + 35 = 1, 20$, and $3, 21 - 50 = 2, 31$. Multiplication is worked similarly; for example, $2 \cdot 16 = 32$ and $30 \cdot 2 = 1, 0$. Compute the following:

(a) $3, 20 + 45, 54$

(b) $34, 19, 56 + 26, 0, 4$

(c) $34, 19, 56 - 26, 0, 4$

(d) $44, 0, 0 - 37, 37, 37$

(e) $39, 13 \times 2, 0$

(f) $3, 20 \times 45, 54$

5. In sexagesimal the notation $0; 13, 54 = 13 \cdot 60^{-1} + 54 \cdot 60^{-2}$.

(a) Convert .365 to sexagesimal.

(b) The ratio 1 : 2 is the number 0; 30 in sexagesimal, and 1 : 8 is 0; 7, 30. Explain how this is computed and use the found algorithm to calculate 1 : 15 and 1 : 50, and 1 : 21.

(c) Find the sexagesimal numeral for 3 : 8.

6. Explain why the area of a regular polygon equals one half the product of its apothem and perimeter.

7. Find the area of a square whose apothem is 10.

8. Confirm that the ratio of the area of a regular hexagon and the square of one of its sides is approximately $2\frac{5}{8}$.

9. Find the ratio of the perimeter of a regular pentagon to the circumference of its circumscribed circle.

1.3 CHINA

The Han dynasty witnessed a rapid increase in science and technology. Agricultural production also saw an increase during this time. To handle the logistics of this larger endeavor and to maximize the chances of successful crops, the Chinese believed it was crucial to have accurate calendars and reliable weather forecasts. This led to improved methods in astronomy and the computational skills that went along with it, yet the mathematics upon which their science rested was the culmination of some 800 years of development dating back to the Zhou and Qin dynasties. Despite the long lineage, it was during the middle of the Former Han dynasty that the earliest extant Chinese text containing mathematics was written. Its title is the *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*, and it is primarily a book on astronomy. At the beginning of the work the author writes that the astronomers of the Zhou period would erect vertical stakes in the ground to track the motion of the sun. We call such a stake a **gnomon** (Figure 1.8). It enabled the astronomers to perform various celestial calculations. For this reason the gnomon was also known in the West. The Babylonians used it, and from there it was introduced to the Greeks (Yǎn and Shírán 1987, 25, 27; Heath 1921a, 78).

The dating of the mathematics of China is very difficult. A number of the emperors issued book-burning decrees to try to eliminate the memory of past rivals. The first of these was by the tyrant Shih Huang-ti in 213 B.C. Scholars of later dynasties would then attempt to recreate the old works by memory. Another contributing factor to the difficulty of dating is due to scribes attributing new results to past famous scholars, so it is difficult to attribute any given result accurately. In addition the texts were written originally on bamboo and later on silk, making the documents susceptible to decay (Swetz and Kao 1977, 17; Burton 1985, 27). Therefore, it was difficult for a work to survive for long periods of time. The earliest text dedicated solely to mathematics that did endure goes by the name *Nine Chapters on the Mathematical Art*. It presents a complete system of mathematics that served as the foundational work for all later Chinese mathematics. The text is written as a sequence of problems each followed by its solution. As its name implies, it is comprised of nine chapters:

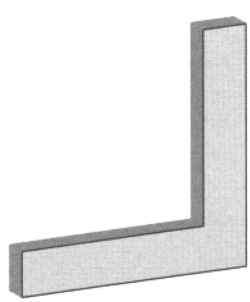


Figure 1.8 A gnomon

1. "Field measurement" shows how to calculate areas.
2. "Cereals" gives problems on proportion.
3. "Distribution by proportion" explains how to allocate goods according to a given proportional distribution.
4. In "What width?" solutions are found for problems where the length of a side is sought when an area or a volume is given. This chapter includes methods on computing square and cube roots.
5. "Construction consultations" focuses on calculating the volume of various solids.
6. "Fair taxes" describes how best to distribute grain and labor according to population and distance.
7. "Excess and deficiency" deals with solving a specific type of system of two equations in two unknowns.
8. A more general presentation of systems of linear equations is given in "Rectangular arrays." How to work with positive and negative numbers is also discussed.
9. "Gōugǔ" discusses right triangles and includes problems involving similar right triangles. An introduction to quadratic equations is also given.

The *Nine Chapters* was probably the work of several mathematicians covering centuries of work (Yǎn and Shírán 1987, 33–35).

Formulas

Although the *Nine Chapters*' main emphasis is algebra, it did include some geometry including an extensive list of formulas for area and volume. Like the Egyptians and Babylonians, the Chinese discovered their formulas empirically. They did not establish them based on reasoning from first principles (Swetz and Kao 1977, 62). Some of their formulas include:

- The area of a circle,
$$A = \frac{C}{2} \cdot \frac{d}{2},$$
where C is the circumference and d is the diameter.
- The area of a ring [Figure 1.9(a)],
$$A = \frac{1}{2}(C_1 + C_2)\Delta r,$$
where C_1 is the outer circumference, C_2 is the inner circumference, and Δr is the difference of the radii.

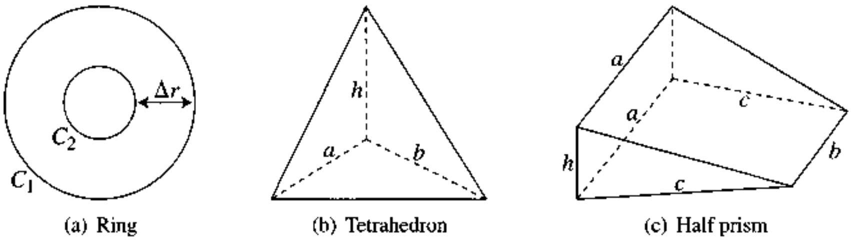


Figure 1.9 An area and two volumes.

- The volume of a tetrahedron with a base consisting of a right triangle [Figure 1.9(b)],

$$V = \frac{1}{6}abh,$$

where its height is h and the legs of its base have length a and b.

- A half-prism with an isosceles trapezoidal base [Figure 1.9(c)],

$$V = \frac{1}{6}(2b + a)ch,$$

where the bases of the trapezoidal base are a and b, the sides of the base have length c, and the height of the half-prism is h.

- The volume of a right circular cylinder,

$$V = \frac{1}{12}C^2h,$$

where C is the circumference of its base, h is its height, and $\pi = 3$.

- The volume of a sphere of diameter d,

$$V = \frac{9}{16}d^3,$$

where again $\pi = 3$.

The accuracy of these formulas is checked in Exercise 1.

Gōugǔ Theorem

The *Arithmetical Classic* emphasizes that mastery in calculation is only the first step in learning mathematics. Taking these basic tools and applying them in other situations is the more important step. In a dialogue that is recorded in the *Arithmetical Classic*, the master tells his student that "by asking one question one can reach ten thousand things." An example of a basic result that leads to many applications is

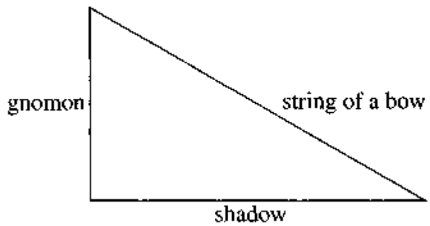


Figure 1.10 The triangle for the Gōugǔ Theorem.

the **Gōugǔ Theorem**. Take a right triangle. Call the vertical leg the *gnomon*, the horizontal leg the *shadow*, and the hypotenuse the *string of a bow* (Figure 1.10). The Gōugǔ Theorem allows us to conclude that

$$(\text{gnomon})^2 + (\text{shadow})^2 = (\text{string of a bow})^2,$$

and the *Arithmetical Classic* notes the 3-4-5 triangle as a special case. The Chinese probably mastered the Gōugǔ Theorem early because it gained almost legendary status. As with many of the significant mathematical works in ancient times, people would write **commentaries** in which they would clarify and expand on what was written. Sometimes the commentator would include tales that emphasized the importance of a result. An example appears in the *Arithmetical Classic* concerning the Gōugǔ Theorem. The author writes:

> Emperor Yǔ quells floods, he deepens rivers and streams, observes the shape of mountains and valleys, surveys the high and low places, relieves the greatest calamities and saves the people from danger. He leads the floods east into the sea and ensures no flooding or drowning. This is made possible because of the Gōugǔ theorem. . . . (Yǎn and Shírán 1987, 29–30)

A well-known problem that requires the Gōugǔ Theorem we will call the **broken bamboo problem** (Swetz and Kao 1977, 44–45). It is illustrated in Figure 1.11.

> A bamboo shoot stands 10 feet tall and perpendicular to the ground. There is a break in the shoot near the top that causes a bend in the shoot that allows the top to touch the ground 3 feet from the base of the bamboo. What is the length of the stem that is left standing erect?

This problem is also found in an Indian work, *Compendium of Calculation*. It was written by Mahavira in the ninth century. It also appeared in Philippi Calandri's *Arithmetic*. To solve it, let x be the height of the standing shoot, and then $10 - x$ is the length of the bent portion. We shall follow the traditional solution. From the figure we see that

$$9 = (10 - x)^2 - x^2 = (10 - x + x)(10 - x - x) = 10(10 - 2x).$$

Therefore,

$$\tfrac{9}{10} = 10 - 2x,$$

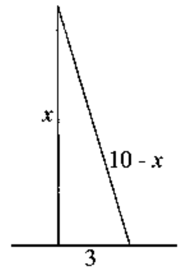


Figure 1.11 The broken bamboo problem.

from which we find that

$$x = 4\tfrac{11}{20}.$$

The fame of the Gōugǔ Theorem extended to its proof, which had its own name. It was called ***hsuan-thu***, and it is the oldest known proof of what we know as the Pythagorean Theorem. It is illustrated in Figure 1.12. Begin by taking a 3×4 rectangle like the one in the lower left-hand corner of the diagram. Its diagonal is 5 units long, so the area of the square constructed on this diagonal is 25 square units. Construct three more rectangles identical in dimension to the original rectangle and place them adjacent to each other, forming a 7×7 square. Its area is

$$(3 + 4)^2 = 49 \text{ square units.}$$

The shaded square cuts each of the four rectangles in half. These four halves can be joined to form two 3×4 rectangles that have total area of

$$2 \cdot 3 \cdot 4 = 24 \text{ square units.}$$

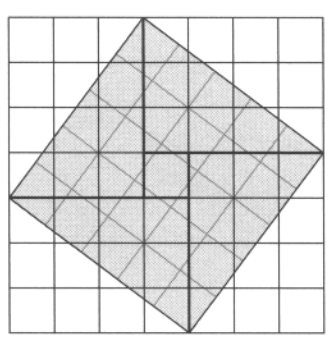


Figure 1.12 The *hsuan-thu*.

When the area of these two rectangles is subtracted from the area of the large square, the remaining region is the shaded square. This illustrates that

$$(3+4)^2 - 2 \cdot 3 \cdot 4 = 5^2,$$

that is,

$$3^2 + 2 \cdot 3 \cdot 4 + 4^2 - 2 \cdot 3 \cdot 4 = 5^2,$$

and then

$$3^2 + 4^2 = 5^2.$$

The *hsuan-thu* inspired the Hindu mathematician Bhaskara. He was born in India in A.D. 1114. For his proof he took a right triangle with legs of length a and b and hypotenuse c. He made three copies of the triangle and arranged all four into the shape of a square. The triangles are arranged such that in the middle of the large square is a smaller one with sides of length $a - b$ [Figure 1.13(a)]. What Bhaskara did next was take the five pieces of the large square and rearrange them so that he formed a shape that had area equal to $a^2 + b^2$ [Figure 1.13(b)]. We can imagine that he was so delighted with his proof that the only thing left for him to say was "Behold!" (Swetz and Kao 1977, 13; Burton 1985, 114).

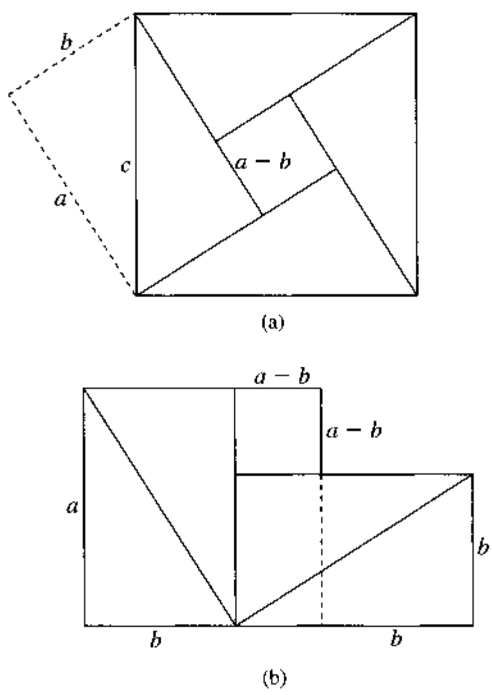


Figure 1.13 Behold!

Exercises

1. Check the accuracy of the area and volume formulas that start on page 21.

2. The following is a sample of right-triangle problems from the *Nine Chapters* (Swetz and Kao 1977, 26f).

   (a) A 7-inch-thick board is to be cut from a wooden log 2-feet, 5-inches in diameter. What is the maximum width of the board?

   (b) A 20-foot-tall tree has a circumference of 3 feet. A vine winds around the tree seven times until it reaches the top, forming a perfect helix. What is the length of the vine?

   (c) In the center of a square pond with sides of length 10 feet grows a reed. The top of the reed is 1-foot above the surface of the water, and if it is pulled toward the edge of the pond, its top will be even with the water's surface. Find the total height of the reed and the depth of the pond.

   (d) The height of a wall is 10 feet, and a pole leans against it so that the pole reaches the top of the wall. If the bottom of the pole is moved 1 foot farther from the base of the wall, the pole will fall. What is the height of the pole?

   (e) When we use a rod to measure a doorway, we find that the rod is 4 feet longer than the width of the door, 2 feet longer than the height, and the same length as the diagonal. What are the dimensions of the door?

   (f) What is the radius of a circle inscribed within a right triangle with legs of lengths 8 and 15 feet?

   (g) There is a square walled city with a door at the center of each of its four sides. Find the dimensions of the city if there is a tree that is 30 yards due north of the northern gate that can just be seen beyond the corner of the city if one stands 750 yards due west of the western gate.

3. Use dots aligned in the shape of gnomons to show that

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

4. The *Nine Chapters* illustrated a "patchwork" method for calculating volumes (Yǎn and Shírán 1987, 74–75). Use any method to find the volume of the *chú méng* with height h and an $a + 2b$ by $2c$ base.