
1

INTRODUCTION AND SOME USEFUL REVIEW

1.1 A MESSAGE FOR THE STUDENT

This is an advanced-level book based on a course sequence taught by the author for more than 20 years. Prior exposure to transport phenomena is assumed and familiarity with the classic, *Transport Phenomena*, 2nd edition, by R. B. Bird, W. E. Stewart, and E. N. Lightfoot (BS&L), will prove particularly advantageous because the notation adopted here is mainly consistent with BS&L.

There are *many* well-written and useful texts and monographs that treat aspects of transport phenomena. A few of the books that I have found to be especially valuable for engineering problem solving are listed here:

Transport Phenomena, 2nd edition, Bird, Stewart, and Lightfoot.

An Introduction to Fluid Dynamics and *An Introduction to Mass and Heat Transfer*, Middleman.

Elements of Transport Phenomena, Sissom and Pitts.

Transport Analysis, Hershey.

Analysis of Transport Phenomena, Deen.

Transport Phenomena Fundamentals, Plawsky.

Advanced Transport Phenomena, Slattery.

Advanced Transport Phenomena: Fluid Mechanics and Convective Transport Processes, Leal.

The Phenomena of Fluid Motions, Brodkey.

Fundamentals of Heat and Mass Transfer, Incropera and De Witt.

Fluid Dynamics and Heat Transfer, Knudsen and Katz.

Fundamentals of Momentum, Heat, and Mass Transfer, 4th edition, Welty, Wicks, Wilson, and Rorrer.

Fluid Mechanics for Chemical Engineers, 2nd edition, Wilkes.

Vectors, Tensors, and the Basic Equations of Fluid Mechanics, Aris.

In addition, there are many other more specialized works that treat or touch upon some facet of transport phenomena. These books can be very useful in proper circumstances and they will be clearly indicated in portions of this book to follow. In view of this sea of information, what is the point of yet another book? Let me try to provide my rationale below.

I taught transport phenomena for the first time in 1977–1978. In the 30 years that have passed, I have taught our graduate course sequence, Advanced Transport Phenomena 1 and 2, more than 20 times. These experiences have convinced me that no suitable *single* text exists in this niche, hence, this book.

So, the course of study you are about to begin here *is the course sequence* I provide for our first-year graduate students. It is important to note that for many of our students, *formal exposure* to fluid mechanics and heat transfer *ends with this course sequence*. It is imperative that such students leave the experience with, at the very least, some cognizance of the breadth of transport phenomena. Of course, this reality has profoundly influenced this text.

In 1982, I purchased my first IBM PC (personal computer); by today's standards it was a *kludge* with a very low clock rate,

just 64K memory, and 5.25" (160K) floppy drives. The high-level language available at that time was interpreted BASIC that had severe limits of its own with respect to execution speed and array size. Nevertheless, it was immediately apparent that the decentralization of computing power would spur a revolution in engineering problem solving. By necessity I became fairly adept at BASIC programming, first using the interpreter and later using various BASIC compilers. Since 1982, the increases in PC capability and the decreases in cost have been astonishing; it now appears that Moore's "law" (*the number of transistors on an integrated circuit yielding minimum component cost doubles every 24 months*) may continue to hold true through several more generations of chip development. In addition, PC hard-drive capacity has exhibited exponential growth over that time frame and the estimated cost per G-FLOP has decreased by a factor of about 3 every year for the past decade.

It is not an exaggeration to say that a cheap desktop PC in 2009 has much more computing power than a typical university mainframe computer of 1970. As a consequence, problems that were pedagogically impractical are now routine. This computational revolution has changed the way I approach instruction in transport phenomena and it has made it possible to assign more complex exercises, even embracing nonlinear problems, and still maintain expectations of timely turnaround of student work. It was my intent that this computational revolution be reflected in this text and in some of the problems that accompany it. However, I have avoided significant use of *commercial software* for problem solutions.

Many engineering educators have come to the realization that computers (and the microelectronics revolution in general) are changing the way students learn. The ease with which complicated information can be obtained and difficult problems can be solved has led to a physical disconnect; students have far fewer opportunities to develop somatic comprehension of problems and problem solving in this new environment. The reduced opportunity to experience has led to a reduced ability to perceive, and with dreadful consequence. Recently, Haim Baruh (2001) observed that the computer revolution has led young people to "think, learn and visualize differently. . . . Because information can be found so easily and quickly, students often skip over the basics. For the most part, abstract concepts that require deeper thought aren't part of the equation. I am concerned that unless we use computers wisely, the decline in student performance will continue."

Engineering educators must remember that computers are merely tools and skillful use of a commercial software package does not translate to the type of understanding needed for the formulation and analysis of engineering problems. In this regard, I normally ask students to be wary of reliance upon commercial software for solution of problems in transport phenomena. In certain cases, commercial codes can be used for comparison of alternative models; this is particularly useful *if* the software can be verified with *known* results for

that particular scenario. *But*, blind acceptance of black-box computations for an untested situation is foolhardy.

One of my principal objectives in transport phenomena instruction is to help the student develop physical insight and problem-solving capability simultaneously. This balance is essential because either skill set alone is just about useless. In this connection, we would do well to remember G. K. Batchelor's (1967) admonition: "By one means or another, a teacher should show the relation between his analysis and the behavior of real fluids; fluid dynamics is much less interesting if it is treated largely as an exercise in mathematics." I also feel strongly that how and why this field of study developed is not merely peripheral; one can learn a great deal by obtaining a historical perspective and in many instances I have tried to provide this. I believe in the adage that you cannot know where you are going if you do not know where you have been. Many of the accompanying problems have been developed to provide a broader view of transport phenomena as well; they constitute a unique feature of this book, and many of them require the student to draw upon other resources.

I have tried to recall questions that arose in my mind when I was beginning my second course of study of transport phenomena. I certainly hope that some of these have been clearly treated here. For many of the examples used in this book, I have provided details that might often be omitted, but this has a price; the resulting work cannot be as broad as one might like. There are some important topics in transport phenomena that are not treated in a substantive way in this book. These omissions include non-Newtonian rheology and energy transport by radiation. Both topics deserve far more consideration than could be given here; fortunately, both are subjects of numerous specialized monographs. In addition, both boundary-layer theory and turbulence could easily be taught as separate one- or even two-semester courses. That is obviously not possible within our framework. I would like to conclude this message with five observations:

1. Transport phenomena are pervasive and they impact upon *every* aspect of life.
2. Rote learning is ineffective in this subject area because the successful application of transport phenomena is directly tied to physical understanding.
3. Mastery of this subject will enable you to critically evaluate many physical phenomena, processes, and systems across many disciplines.
4. Student effort is paramount in graduate education. There are many places in this text where outside reading and additional study are not merely recommended, but expected.
5. Time has not diminished my interest in transport phenomena, and my hope is that through this book I can share my enthusiasm with students.

1.2 DIFFERENTIAL EQUATIONS

Students come to this sequence of courses with diverse mathematical backgrounds. Some do not have the required levels of proficiency, and since these skills are crucial to success, a brief review of some important topics may be useful.

Transport phenomena are governed by, and modeled with, differential equations. These equations may arise through mass balances, momentum balances, and energy balances. The main equations of change are second-order partial differential equations that are (too) frequently nonlinear. One of our principal tasks in this course is to find solutions for such equations; we can expect this process to be challenging at times.

Let us begin this section with some simple examples of ordinary differential equations (ODEs); consider

$$\frac{dy}{dx} = c \quad (c \text{ is constant}) \quad (1.1)$$

and

$$\frac{dy}{dx} = y. \quad (1.2)$$

Both are *linear*, first-order ordinary differential equations. Remember that linearity is determined by the dependent variable y . The solutions for (1.1) and (1.2) are

$$y = cx + C_1 \text{ and } y = C_1 \exp(x), \text{ respectively.} \quad (1.3)$$

Note that if $y(x=0)$ is specified, then the behavior of y is set for all values of x . If the independent variable x were time t , then the future behavior of the system would be known. This is what we mean when we say that a system is deterministic. Now, what happens when we modify (1.2) such that

$$\frac{dy}{dx} = 2xy? \quad (1.4)$$

We find that $y = C_1 \exp(x^2)$. These first-order linear ODEs have all been separable, admitting simple solution. We will sketch the (three) behaviors for $y(x)$ on the interval 0–2, given that $y(0) = 1$ (Figure 1.1). *Match each of the three curves with the appropriate equation.*

Note what happens to $y(x)$ if we continue to add additional powers of x to the right-hand side of (1.4), allowing y to remain. If we add powers of y instead—and make the equation inhomogeneous—we can expect to work a little harder. Consider this first-order *nonlinear* ODE:

$$\frac{dy}{dx} = a + by^2. \quad (1.5)$$

This is a type of Riccati equation (Jacopo Francesco Count Riccati, 1676–1754) and the nature of the solution will

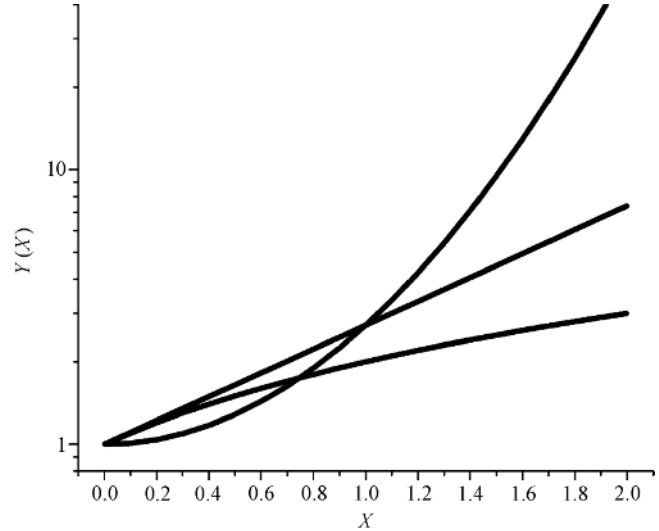


FIGURE 1.1. Solutions for $dy/dx = 1$, $dy/dx = y$, and $dy/dx = 2xy$.

depend on the product of a and b . If we let $a = b = 1$, then

$$y = \tan(x + C_1). \quad (1.6)$$

Before we press forward, we note that Riccati equations were studied by Euler, Liouville, and the Bernoulli's (Johann and Daniel), among others. How will the solution change if eq. (1.5) is rewritten as

$$\frac{dy}{dx} = 1 - y^2? \quad (1.7)$$

Of course, the equation is still separable, so we can write

$$\int \frac{dy}{1 - y^2} = x + C_1. \quad (1.8)$$

Show that the solution of (1.8), given that $y(0) = 0$, is $y = \tanh(x)$.

When a first-order differential equation arises in transport phenomena, it is usually by way of a macroscopic balance, for example, $[\text{Rate in}] - [\text{Rate out}] = [\text{Accumulation}]$. Consider a 55-gallon drum (vented) filled with water. At $t = 0$, a small hole is punched through the side near the bottom and the liquid begins to drain from the tank. If we let the velocity of the fluid through the orifice be represented by Torricelli's theorem (a frictionless result), a mass balance reveals

$$\frac{dh}{dt} = -\frac{R_0^2}{R_T^2} \sqrt{2gh}, \quad (1.9)$$

where R_0 is the radius of the hole. This equation is easily solved as

$$h = \left[-\sqrt{\frac{g}{2}} \frac{R_0^2}{R_T^2} t + C_1 \right]^2. \quad (1.10)$$

The drum is initially full, so $h(t=0) = 85$ cm and $C_1 = 9.21954$ cm^{1/2}. Since the drum diameter is about 56 cm, $R_T = 28$ cm; if the radius of the hole is 0.5 cm, it will take about 382 s for half of the liquid to flow out and about 893 s for 90% of the fluid to escape. If friction is taken into account, how would (1.9) be changed, and how much more slowly would the drum drain?

We now contemplate an increase in the *order* of the differential equation. Suppose we have

$$\frac{d^2 y}{dx^2} + a = 0, \quad (1.11)$$

where a is a constant or an elementary function of x . This is a common equation type in transport phenomena for steady-state conditions with molecular transport occurring in one direction. We can immediately write

$$\begin{aligned} \frac{dy}{dx} &= -\int a \, dx + C_1, \text{ and if } a \text{ is a constant,} \\ y &= -\frac{a}{2}x^2 + C_1x + C_2. \end{aligned}$$

Give an example of a specific type of problem that produces this solution. One of the striking features of (1.11) is the absence of a first derivative term. You might consider what conditions would be needed in, say, a force balance to produce *both* first and second derivatives.

The simplest second-order ODEs (that include first derivatives) are linear equations with constant coefficients. Consider

$$\frac{d^2 y}{dx^2} + 1 \frac{dy}{dx} + y = f(x), \quad (1.12)$$

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = f(x), \quad (1.13)$$

and

$$\frac{d^2 y}{dx^2} + 3 \frac{dy}{dx} + y = f(x). \quad (1.14)$$

Using linear differential operator notation, we rewrite the left-hand side of each and factor the result:

$$(D^2 + D + 1)y \quad \left(D + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(D + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right), \quad (1.15)$$

$$(D^2 + 2D + 1)y \quad (D + 1)(D + 1), \quad (1.16)$$

$$(D^2 + 3D + 1)y \quad \left(D + \frac{3 + \sqrt{5}}{2}\right)\left(D + \frac{3 - \sqrt{5}}{2}\right). \quad (1.17)$$

Now suppose the forcing function $f(x)$ in (1.12)–(1.14) is a constant, say 1. What do (1.15)–(1.17) tell you about the nature of possible solutions? The complex conjugate roots in (1.15) will result in oscillatory behavior. Note that all three of these second-order differential equations have constant coefficients and a first derivative term. If eq. (1.14) had been developed by force balance (with x replaced by t), the dy/dx (velocity) term might be some kind of frictional resistance. We do not have to expend much effort to find second-order ODEs that pose greater challenges. What if you needed a solution for the nonlinear equation

$$\frac{d^2 y}{dx^2} = a + by + cy^2 + dy^3? \quad (1.18)$$

Actually, a number of closely related equations have figured prominently in physics. Einstein, in an investigation of planetary motion, was led to consider

$$\frac{d^2 y}{dx^2} + y = a + by^2. \quad (1.19)$$

Duffing, in an investigation of forced vibrations, carried out a study of the equation

$$\frac{d^2 y}{dx^2} + k \frac{dy}{dx} + ay + by^3 = f(x). \quad (1.20)$$

A limited number of nonlinear, second-order differential equations can be solved with (Jacobian) elliptic functions. For example, Davis (1962) shows that the solution of the nonlinear equation

$$\frac{d^2 y}{dx^2} = 6y^2 \quad (1.21)$$

can be written as

$$y = A + \frac{B}{\text{sn}^2(C(x - x_1))}. \quad (1.22)$$

Tabulated values are available for the Jacobi elliptic sine, sn ; see pages 175–176 in Davis (1962). The reader desiring an introduction to elliptic functions is encouraged to work problem 1.N in this text, read Chapter 5 in Vaughn (2007), and consult the extremely useful book by Milne-Thomson (1950).

The point of the immediately preceding discussion is as follows: The elementary functions that are familiar to us, such

as sine, cosine, exp, ln, etc., are solutions to linear differential equations. Furthermore, when constants arise in the solution of linear differential equations, they do so linearly; for an example, see the solution of eq. (1.11) above. In nonlinear differential equations, arbitrary constants appear *nonlinearly*. Nonlinear problems abound in transport phenomena and we can expect to find analytic solutions only for a very limited number of them. Consequently, most nonlinear problems must be solved numerically and this raises a host of other issues, including existence, uniqueness, and stability.

So much of our early mathematical education is bound to linearity that it is difficult for most of us to perceive and appreciate the beauty (and beastliness) in nonlinear equations. We can illustrate some of these concerns by examining the elementary nonlinear difference (logistic) equation,

$$X_{n+1} = \alpha X_n(1 - X_n). \quad (1.23)$$

Let the parameter α assume an initial value of about 3.5 and let $X_1 = 0.5$. Calculate the new value of X and insert it on the right-hand side. As we repeat this procedure, the following sequence emerges: 0.5, 0.875, 0.38281, 0.82693, 0.5009, 0.875, 0.38282, 0.82694, . . . Now allow α to assume a slightly larger value, say 3.575. Then, the sequence of calculated values is 0.5, 0.89375, 0.33949, 0.80164, 0.56847, 0.87699, 0.38567, 0.84702, 0.46324, 0.88892, 0.353, 0.8165, 0.53563, 0.88921, 0.35219, 0.81564, 0.53757, . . . We can continue this process and report these results graphically; the result is a bifurcation diagram. How would you characterize Figure 1.2? Would you be tempted to use “chaotic” as a descriptor? The most striking feature of this logistic map is that a completely deterministic equation produces behavior that superficially appears to be random (*it is not*). Baker and

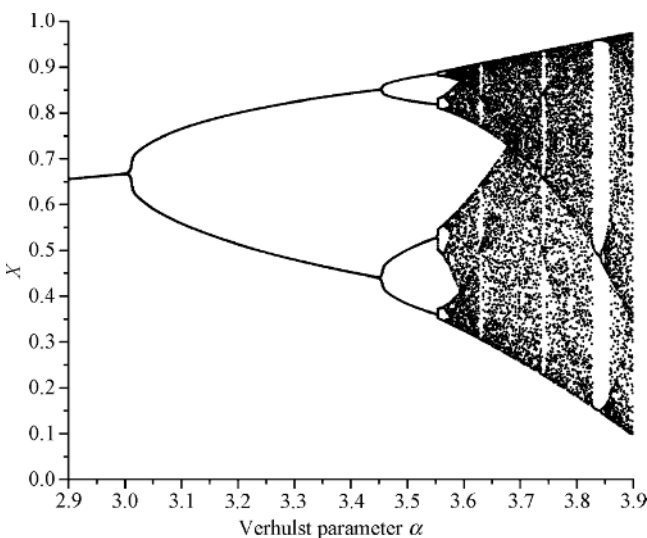


FIGURE 1.2. Bifurcation diagram for the logistic equation with the Verhulst parameter α ranging from 2.9 to 3.9.

Gollub (1990) described this map as having regions where the behavior is chaotic with windows of periodicity.

Note that the chaotic behavior seen above is attained through a series of period doublings (or pitchfork bifurcations). Baker and Gollub note that many dynamical systems exhibit this path to chaos. In 1975, Mitchell Feigenbaum began to look at period doublings for a variety of rather simple functions. He quickly discovered that all of them had a common characteristic, a universality; that is, the ratio of the spacings between successive bifurcations was always the same:

$$4.6692016 \dots \quad (\text{Feigenbaum number}).$$

This leads us to hope that a relatively simple system or function might serve as a model (or at least a surrogate) for far more complex behavior.

We shall complete this part of our discussion by selecting two terms from the x -component of the Navier–Stokes equation,

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + \dots, \quad (1.24)$$

and writing them in finite difference form, letting i be the spatial index and j the temporal one. We can drop the subscript “ x ” for convenience. One of the possibilities (though not a very good one) is

$$\frac{v_{i,j+1} - v_{i,j}}{\Delta t} + v_{i,j} \frac{v_{i+1,j} - v_{i,j}}{\Delta x} + \dots \quad (1.25)$$

We might imagine this being rewritten as an explicit algorithm (where we calculate v at the new time, $j + 1$, using velocities from the j th time step) in the following form:

$$v_{i,j+1} \approx v_{i,j} - \frac{\Delta t}{\Delta x} v_{i,j} (v_{i+1,j} - v_{i,j}) + \dots \quad (1.26)$$

Please make note of the dimensionless quantity $\Delta t v_{i,j} / \Delta x$; this is the Courant number, Co , and it will be extremely important to us later. As a computational scheme, eq. (1.26) is generally unworkable, but note the similarity to the logistic equation above. The nonlinear character of the equations that govern fluid motion guarantees that we will see unexpected beauty and maddening complexity, if only we knew where (and how) to look.

In this connection, a system that evolves in time can often be usefully studied using phase space analysis, which is an underutilized tool for the study of the dynamics of low-dimension systems. Consider a periodic function such as $f(t) = A \sin(\omega t)$. The derivative of this function is $\omega A \cos(\omega t)$. If we cross-plot $f(t)$ and df/dt , we will obtain a *limit cycle* in the shape of an ellipse. That is, the system trajectory in phase space takes the form of a closed path, which is expected

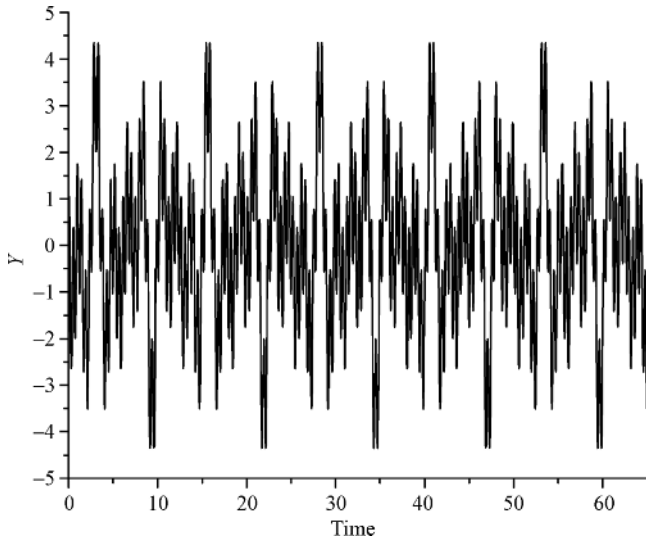


FIGURE 1.3. “Artificial” time-series data constructed from sinusoids.

behavior for a purely periodic function. If, on the other hand, we had an oscillatory system that was unstable, the amplitude of the oscillations would grow in time; the resulting phase-plane portrait would be an outward spiral. An attenuated (damped) oscillation would produce an inward spiral. This technique can be useful for more complicated functions or signals as well. Consider the oscillatory behavior illustrated in Figure 1.3.

If you look closely at this figure, you can see that the function $f(t)$ does exhibit periodic behavior—many features of the system output appear repeatedly. In phase space, this system yields the trajectory shown in Figure 1.4.

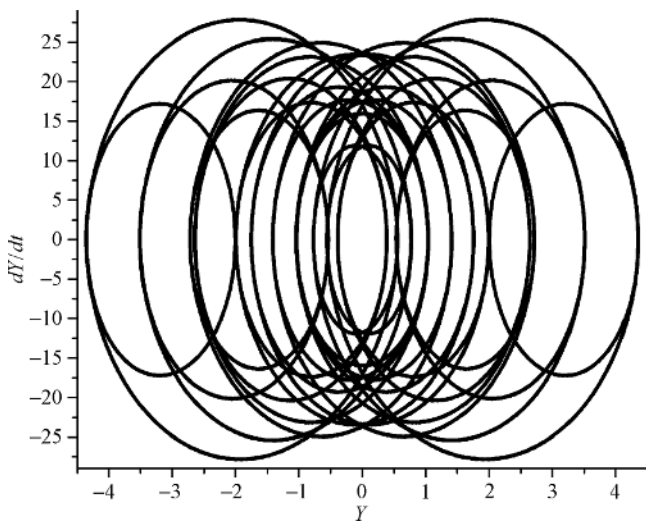


FIGURE 1.4. Phase space portrait of the system dynamics illustrated in Figure 1.3.

What we see here is the combination of a limited number of periodic functions interacting. Particular points in phase space are revisited fairly regularly. But, if the dynamic behavior of a system was truly chaotic, we might see a phase space in which *no point is ever revisited*. The implications for the behavior of a perturbed complex nonlinear system, such as the global climate, are sobering.

Another consequence of nonlinearity is sensitivity to initial conditions; to solve a general fluid flow problem, we would need to consider three components of the Navier–Stokes equation and the continuity relation simultaneously. Imagine an integration scheme forward marching in time. It would be necessary to specify initial values for v_x , v_y , v_z , and p . Suppose that v_x had the exact initial value, 5 cm/s, but your computer represented the number as 4.99999... cm/s. Would the integration scheme evolve along the “correct” pathway? Possibly not. Jules-Henri Poincaré (who was perhaps the last man to understand all of the mathematics of his era) noted in 1908 that “... small differences in the initial conditions produce very great ones in the final phenomena.” In more recent years, this concept has become popularly known as the “butterfly effect” in deference to Edward Lorenz (1963) who observed that the disturbance caused by a butterfly’s wing might change the weather pattern for an entire hemisphere. This is an idea that is unfamiliar to most of us; in much of the educational process we are conditioned to believe a model for a system (a differential equation), taken together with its present state, completely set the future behavior of the system.

Let us conclude this section with an appropriate example; we will explore the Rossler (1976) problem that consists of the following set of three (deceptively simple) ordinary differential equations:

$$\begin{aligned} \frac{dX}{dt} &= -Y - Z, & \frac{dY}{dt} &= X + 0.2Y, & \text{and} \\ \frac{dZ}{dt} &= 0.2 + Z(X - 5.7). \end{aligned} \tag{1.27}$$

Note that there is but one nonlinearity in the set, the product ZX . The Rossler model is synthetic in the sense that it is an abridgement of the Lorenz model of local climate; consequently, it does not have a direct physical basis. But it will reveal some unexpected and important behavior. Our plan is to solve these equations numerically using the initial values of 0, -6.78 , and 0.02 for X , Y , and Z , respectively. We will look at the evolution of all three dependent variables with time, and then we will examine a segment or cut from the system trajectory by cross-plotting X and Y .

The main point to take from this example is that an elementary, low-dimensional system can exhibit unexpectedly complicated behavior. The system trajectory seen in Figure 1.5b is a portrait of what is now referred to in the literature as a “strange” attractor. The interested student is encouraged to read the papers by Rossler (1976) and Packard

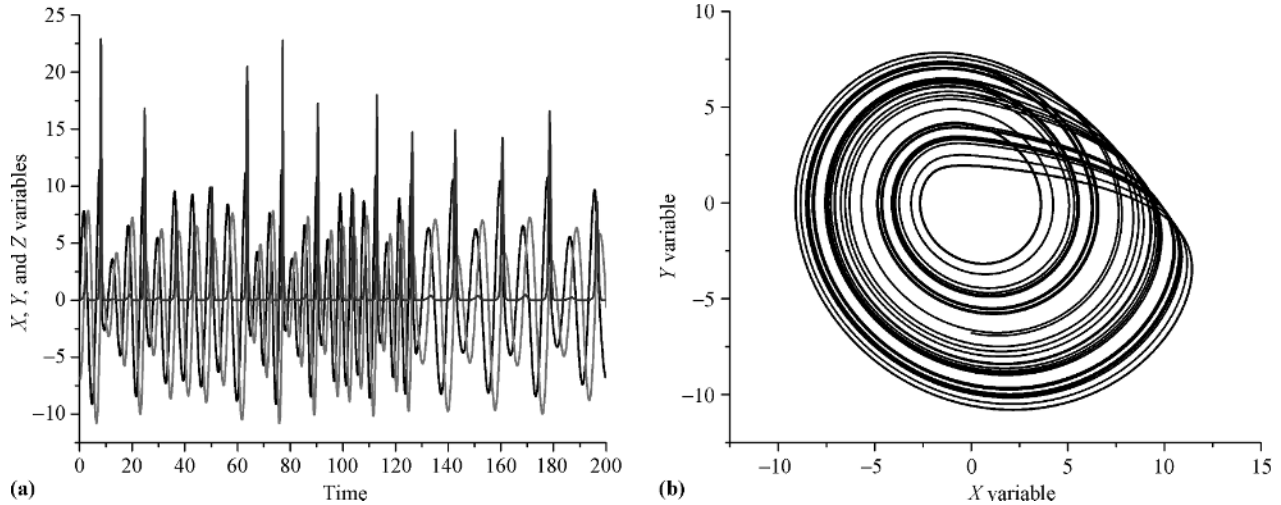


FIGURE 1.5. The Rossler model: $X(t)$, $Y(t)$, and $Z(t)$ for $0 < t < 200$ (a), and a cut from the system trajectory (Y plotted against X) (b).

et al. (1980). The formalized study of chaotic behavior is still in its infancy, but it has become clear that there are applications in hydrodynamics, mechanics, chemistry, etc.

There are additional tools that can be used to determine whether a particular system's behavior is periodic, aperiodic, or chaotic. For example, the rate of divergence of a chaotic trajectory about an attractor is characterized with Lyapunov exponents. Baker and Gollub (1990) describe how the exponents are computed in Chapter 5 of their book and they include a listing of a BASIC program for this task. The Fourier transform is also invaluable in efforts to identify important periodicities in the behavior of nonlinear systems. We will make extensive use of the Fourier transform in our consideration of turbulent flows.

The student with further interest in this broad subject area is also encouraged to read the recent article by Porter et al. (2009). This paper treats a historically significant project carried out at Los Alamos by Fermi, Pasta, and Ulam (Report LA-1940). Fermi, Pasta, and Ulam (FPU) investigated a one-dimensional mass-and-spring problem in which 16, 32, and 64 masses were interconnected with non-Hookean springs. They experimented (computationally) with cases in which the restoring force was proportional to displacement raised to the second or third power(s). FPU found that the nonlinear systems did not share energy (in the expected way) with the higher modes of vibration. Instead, energy was exchanged ultimately among just the first few modes, almost periodically. Since their original intent had been to explore the rate at which the initial energy was distributed among all of the higher modes (they referred to this process as "thermalization"), they quickly realized that the nonlinearities were producing quite unexpectedly localized behavior in phase space! The work of FPU represents one of the very first cases in which extensive computational experiments were performed for nonlinear systems.

1.3 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

We have to be able to recognize and classify partial differential equations to attack them successfully; a book like Powers (1979) can be a valuable ally in this effort. Consider the generalized second-order partial differential equation, where ϕ is the dependent variable and x and y are arbitrary independent variables:

$$A \frac{\partial^2 \phi}{\partial x^2} + B \frac{\partial^2 \phi}{\partial x \partial y} + C \frac{\partial^2 \phi}{\partial y^2} + D \frac{\partial \phi}{\partial x} + E \frac{\partial \phi}{\partial y} + F \phi + G = 0. \quad (1.28)$$

A , B , C , D , E , F , and G can be functions of x and y , but *not* of ϕ . This *linear* partial differential equation can be classified as follows:

$$\begin{aligned} B^2 - 4AC < 0 & \quad (\text{elliptic}), \\ B^2 - 4AC = 0 & \quad (\text{parabolic}), \\ B^2 - 4AC > 0 & \quad (\text{hyperbolic}). \end{aligned}$$

For illustration, we look at the "heat" equation (transient conduction in one spatial dimension):

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2}. \quad (1.29)$$

You can see that $A = \alpha$, $B = 0$, and $C = 0$; the equation is parabolic. Compare this with the governing (Laplace) equation for two-dimensional potential flow (ψ is the stream function):

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \quad (1.30)$$

In this case, $A = 1$ and $C = 1$ while $B = 0$; the equation is elliptic. Next, we consider a vibrating string (the wave equation):

$$\frac{\partial^2 u}{\partial t^2} = s^2 \frac{\partial^2 u}{\partial y^2}. \quad (1.31)$$

Note that $A = 1$ and $C = -s^2$; therefore, $-4AC > 0$ and eq. (1.31) is hyperbolic. In transport phenomena, transient problems with molecular transport only (heat or diffusion equations) will have parabolic character. Equilibrium problems such as steady-state diffusion, conduction, or viscous flow in a duct will be elliptic in nature (phenomena governed by Laplace- or Poisson-type partial differential equations). We will see numerous examples of both in the chapters to come. Hyperbolic equations are common in quantum mechanics and high-speed compressible flows, for example, inviscid supersonic flow about an airfoil. The Navier–Stokes equations that will be so important to us later are of mixed character.

The three most common types of boundary conditions used in transport phenomena are Dirichlet, Neumann, and Robin’s. For Dirichlet boundary conditions, the field variable is specified at the boundary. *Two examples:* In a conduction problem, the temperature at a surface might be fixed (at $y = 0$, $T = T_0$); alternatively, in a viscous fluid flow problem, the velocity at a stationary duct wall would be zero. For Neumann conditions, the flux is specified; for example, for a conduction problem with an insulated wall located at $y = 0$, $(\partial T / \partial y)_{y=0} = 0$. A Robin’s type boundary condition results from equating the fluxes; for example, consider the solid–fluid interface in a heat transfer problem. On the solid side heat is transferred by conduction (Fourier’s law), but on the fluid side of the interface we might have mixed heat transfer processes approximately described by Newton’s “law” of cooling:

$$-k \left(\frac{\partial T}{\partial y} \right)_{y=0} = h(T_0 - T_f). \quad (1.32)$$

We hasten to add that the heat transfer coefficient h that appears in (1.32) is an empirical quantity. The numerical value of h is known only for a small number of cases, usually those in which molecular transport is dominant.

One might think that Newton’s “law” of cooling could not possibly engender controversy. That would be a flawed presumption. Bohren (1991) notes that Newton’s own description of the law as translated from Latin is “if equal times of cooling be taken, the degrees of heat will be in geometrical proportion, and therefore easily found by tables of logarithms.” It is clear from these words that Newton meant that the cooling process would proceed exponentially. Thus, to simply write $q = h(T - T_\infty)$, without qualification, is “incorrect.” On the other hand, if one

uses a lumped-parameter model to describe the cooling of an object, $mC_p(dT/dt) = -hA(T - T_\infty)$, then the oft-cited form does produce an exponential decrease in the object’s temperature in accordance with Newton’s own observation. So, do we have an argument over substance or merely semantics? Perhaps the solution is to exercise greater care when we refer to $q = h(T - T_\infty)$; we should probably call it the defining equation for the heat transfer coefficient h and meticulously avoid calling the expression a “law.”

1.4 NUMERICAL SOLUTIONS FOR PARTIAL DIFFERENTIAL EQUATIONS

Many of the examples of numerical solution of partial differential equations used in this book are based on finite difference methods (FDMs). The reader may be aware that the finite element method (FEM) is widely used in commercial software packages for the same purpose. The FEM is particularly useful for problems with either curved or irregular boundaries and in cases where localized changes require a smaller scale grid for improved resolution. The actual numerical effort required for solution in the two cases is comparable. However, FEM approaches usually employ a separate code (or program) for mesh generation and refinement. I decided not to devote space here to this topic because my intent was to make the solution procedures as general as possible and nearly independent of the computing platform and software. By taking this approach, the student without access to specialized commercial software can still solve many of the problems in the course, in some instances using nothing more complicated than either a spreadsheet or an elementary understanding of any available high-level language.

1.5 VECTORS, TENSORS, AND THE EQUATION OF MOTION

For the discussion that follows, recall that temperature T is a scalar (zero-order, or rank, tensor), velocity V is a vector (first-order tensor), and stress τ is a second-order tensor. Tensor is from the Latin “tensus,” meaning to stretch. We can offer the following, rough, definition of a tensor: It is a generalized quantity or mathematical object that *in three-dimensional space* has 3^n components (where n is the order, or rank, of the tensor). From an engineering perspective, tensors are defined over a continuum and transform according to certain rules. They figure prominently in mechanics (stress and strain) and relativity.

The del operator (∇) in rectangular coordinates is

$$\delta_x \frac{\partial}{\partial x} + \delta_y \frac{\partial}{\partial y} + \delta_z \frac{\partial}{\partial z}. \quad (1.33)$$

For a scalar such as T , ∇T is referred to as the gradient (of the scalar field). So, when we speak of the temperature gradient, we are talking about a vector quantity with both direction and magnitude.

A scalar product can be formed by applying ∇ to the velocity vector:

$$\nabla \cdot V = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}, \quad (1.34)$$

which is the divergence of the velocity, $\text{div}(V)$. The physical meaning should be clear to you: For an incompressible fluid ($\rho = \text{constant}$), conservation of mass requires that $\nabla \cdot V = 0$; in 3-space, if v_x changes with x , the other velocity vector components must accommodate the change (to prevent a net outflow). You may recall that a mass balance for an element of *compressible* fluid reveals that the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0. \quad (1.35a)$$

For a compressible fluid, a net outflow results in a change (decrease) in fluid density. Of course, conservation of mass can be applied in cylindrical and spherical coordinates as well:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \quad (1.35b)$$

and

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\rho v_\theta \sin \theta) \\ + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}(\rho v_\phi) = 0. \end{aligned} \quad (1.35c)$$

In fluid flow, rotation of a suspended particle can be caused by a variation in velocity, even if every fluid element is traveling a path parallel to the confining boundaries. Similarly, the interaction of forces can create a moment that is obtained from the cross product or curl. This tendency toward rotation is particularly significant, so let us review the cross product $\nabla \times V$ in rectangular coordinates:

$$\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \quad (1.36a)$$

$$\nabla \times V = \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \quad (1.36b)$$

$$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad (1.36c)$$

Note that the cross product of vectors is a vector; furthermore, you may recall that (1.36a)–(1.36c), the *vorticity* vector components ω_x , ω_y , and ω_z , are measures of the rate of fluid rotation about the x , y , and z axes, respectively. Vorticity is

extremely useful to us in hydrodynamic calculations because in the interior of a homogeneous fluid vorticity is neither created nor destroyed; it is produced solely at the flow boundaries. Therefore, it often makes sense for us to employ the vorticity transport equation that is obtained by taking the curl of the equation of motion. We will return to this point and explore it more thoroughly later. In cylindrical coordinates, $\nabla \times V$ is

$$\frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \quad (1.37a)$$

$$\nabla \times V = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \quad (1.37b)$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (1.37c)$$

These equations, (1.37a)–(1.37c), correspond to the r , θ , and z components of the vorticity vector, respectively.

The stress tensor τ is a second-order tensor (nine components) that includes both tangential and normal stresses. For example, in rectangular coordinates, τ is

$$\begin{array}{ccc} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{array}$$

The normal stresses have the repeated subscripts and they appear on the diagonal. Please note that the sum of the diagonal components is the *trace* of the tensor (A) and is often written as $\text{tr}(A)$. The trace of the stress tensor, $\Sigma \tau_{ii}$, is *assumed* to be related to the pressure by

$$p = -\frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}). \quad (1.38)$$

Often the pressure in (1.38) is written using the Einstein summation convention as $p = -\tau_{ii}/3$, where the repeated indices imply summation. The shear stresses have differing subscripts and the corresponding off-diagonal terms are equal; that is, $\tau_{xy} = \tau_{yx}$. This requirement is necessary because without it a small element of fluid in a shear field could experience an infinite angular acceleration. Therefore, the stress tensor is symmetric and has just six independent quantities. We will temporarily represent the (shear) stress components by

$$\tau_{ji} = -\mu \frac{\partial v_i}{\partial x_j}. \quad (1.39)$$

Note that this relationship (Newton's law of friction) between stress and strain is linear. There is little *a priori* evidence for its validity; however, known solutions (e.g., for Hagen–Poiseuille flow) are confirmed by physical experience.

It is appropriate for us to take a moment to think a little bit about how a material responds to an applied stress. Strain, denoted by e and referred to as displacement, is often written

as $\Delta l/l$. It is a second-order tensor, which we will write as e_{ij} . We interpret e_{yx} as a shear strain, dy/dx or $\Delta y/\Delta x$. The normal strains, such as e_{xx} , are positive for an element of material that is stretched (extensional strain) and negative for one that is compressed. The summation of the diagonal components, which we will write as e_{ii} , is the volume strain (or dilatation). Thus, when we speak of the ratio of the volume of an element (undergoing deformation) to its initial volume, V/V_0 , we are referring to dilatation. Naturally, dilatation for a real material must lie between zero and infinity. Now consider the response of specific material types; suppose we apply a fixed stress to a material that exhibits Hookean behavior (e.g., by applying an extensional force to a spring). The response is immediate, and when the stress is removed, the material (spring) recovers its initial size. Contrast this with the response of a Newtonian fluid; under a fixed shear stress, the resulting strain rate is constant, and when the stress is removed, the deformation remains. Of course, if a Newtonian fluid is incompressible, no applied stress can change the fluid element's volume; that is, the dilatation is zero. Among "real fluids," there are many that exhibit characteristics of *both* elastic solids and Newtonian fluids. For example, if a viscoelastic material is subjected to constant shear stress, we see some instantaneous deformation that is reversible, followed by flow that is not.

We now sketch the derivation of the equation of motion by making a momentum balance upon a cubic volume element of fluid with sides Δx , Δy , and Δz . We are formulating a vector equation, but it will suffice for us to develop just the x -component. The rate at which momentum accumulates within the volume should be equal to the rate at which momentum enters minus the rate at which momentum leaves (plus the sum of forces acting upon the volume element). Consequently, we write

$$\text{accumulation} \quad \Delta x \Delta y \Delta z \frac{\partial}{\partial t}(\rho v_x) = \quad (1.40a)$$

convective transport of x -momentum in the x -, y -, and z -directions

$$\begin{aligned} & +\Delta y \Delta z v_x \rho v_x|_x - \Delta y \Delta z v_x \rho v_x|_{x+\Delta x} \\ & +\Delta x \Delta z v_y \rho v_x|_y - \Delta x \Delta z v_y \rho v_x|_{y+\Delta y} \\ & +\Delta x \Delta y v_z \rho v_x|_z - \Delta x \Delta y v_z \rho v_x|_{z+\Delta z} \end{aligned} \quad (1.40b)$$

molecular transport of x -momentum in the x -, y -, and z -directions

$$\begin{aligned} & +\Delta y \Delta z \tau_{xx}|_x - \Delta y \Delta z \tau_{xx}|_{x+\Delta x} \\ & +\Delta x \Delta z \tau_{yx}|_y - \Delta x \Delta z \tau_{yx}|_{y+\Delta y} \\ & +\Delta x \Delta y \tau_{zx}|_z - \Delta x \Delta y \tau_{zx}|_{z+\Delta z} \end{aligned} \quad (1.40c)$$

pressure and gravitational forces

$$+\Delta y \Delta z (p|_x - p|_{x+\Delta x}) + \Delta x \Delta y \Delta z \rho g_x \quad (1.40d)$$

We now divide by $\Delta x \Delta y \Delta z$ and take the limits as all three are allowed to approach zero. The result, upon applying the definition of the first derivative, is

$$\begin{aligned} & \frac{\partial \rho v_x}{\partial t} + \frac{\partial}{\partial x} \rho v_x v_x + \frac{\partial}{\partial y} \rho v_y v_x + \frac{\partial}{\partial z} \rho v_z v_x \\ & = -\frac{\partial p}{\partial x} - \frac{\partial \tau_{xx}}{\partial x} - \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial \tau_{zx}}{\partial z} + \rho g_x. \end{aligned} \quad (1.41)$$

This equation of motion can be written more generally in vector form:

$$\frac{\partial}{\partial t}(\rho v) + [\nabla \cdot \rho v v] = -\nabla p - [\nabla \cdot \tau] + \rho g. \quad (1.41a)$$

If Newton's law of friction (1.39) is introduced into (1.41) and if we take both the fluid density and viscosity to be constant, we obtain the x -component of the *Navier–Stokes* equation:

$$\begin{aligned} & \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \\ & = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x. \end{aligned} \quad (1.42)$$

It is useful to review the assumptions employed by Stokes in his derivation in 1845: (1) the fluid is continuous and the stress is no more than a linear function of strain, (2) the fluid is isotropic, and (3) when the fluid is at rest, it must develop a hydrostatic stress distribution that corresponds to the thermodynamic pressure. Consider the implications of (3): When the fluid is in motion, it is not in thermodynamic equilibrium, yet we still describe the pressure with an equation of state. Let us explore this further; we can write the stress tensor as Stokes did in 1845:

$$\tau_{ij} = -p \delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \delta_{ij} \lambda \operatorname{div} V. \quad (1.43)$$

Now suppose we consider the three normal stresses; we will illustrate with just one, τ_{xx} :

$$\tau_{xx} = -p + 2\mu \left(\frac{\partial v_x}{\partial x} \right) + \lambda \operatorname{div} V. \quad (1.44)$$

We add all three together and then divide by $(-)3$, resulting in

$$-\frac{1}{3}(\tau_{xx} + \tau_{yy} + \tau_{zz}) = p - \left(\frac{2\mu + 3\lambda}{3} \right) \operatorname{div} V. \quad (1.45)$$

If we want the mechanical pressure to be *equal* to (negative one-third of) the trace of the stress tensor, then either

$\operatorname{div} V = 0$, or alternatively, $2\mu + 3\lambda = 0$. If the fluid in question is incompressible, then the former is of course valid. But what about the more general case? If $\operatorname{div} V \neq 0$, then it *would be extremely convenient* if $2\mu = -3\lambda$. This is Stokes' hypothesis; it has been the subject of much debate and it is almost certainly wrong except for monotonic gases. Nevertheless, it seems prudent to accept the simplification since as Schlichting (1968) notes, "... the working equations have been subjected to an unusually large number of experimental verifications, even under quite extreme conditions." Landau and Lifshitz (1959) observe that this second coefficient of viscosity (λ) is different in the sense that it is *not* merely a property of the fluid, as it appears to also depend on the frequency (or timescale) of periodic motions (in the fluid). Landau and Lifshitz also state that if a fluid undergoes expansion or contraction, then thermodynamic equilibrium must be restored. They note that if this relaxation occurs slowly, then it is possible that λ is large. There is some evidence that λ may actually be positive for liquids, and the student with deeper interest in Stokes' hypothesis may wish to consult Truesdell (1954).

We can use the substantial time derivative to rewrite eq. (1.42) more compactly:

$$\rho \frac{Dv}{Dt} = -\nabla p + \mu \nabla^2 v + \rho g. \quad (1.46)$$

We should review the meaning of the terms appearing above. On the left-hand side, we have the accumulation of momentum and the convective transport terms (these are the nonlinear inertial terms). On the right-hand side, we have pressure forces, the molecular transport of momentum (viscous friction), and external body forces such as gravity. Please note that the density and the viscosity are assumed to be constant. Consequently, we should identify (1.46) as the Navier–Stokes equation; it is inappropriate to refer to it as the generalized *equation of motion*. We should also observe that for the arbitrary three-dimensional flow of a *nonisothermal, compressible fluid*, it would be necessary to solve (1.41), along with the y - and z -components, the equation of continuity (1.35a), the equation of energy, and an equation of state simultaneously. In this type of problem, the six dependent variables are v_x, v_y, v_z, p, T , and ρ .

As noted previously, we can take the curl of the Navier–Stokes equation and obtain the vorticity transport equation, which is very useful for the solution of some hydrodynamic problems:

$$\frac{\partial \omega}{\partial t} = \nabla \times (v \times \omega) + \nu \nabla^2 \omega, \quad (1.47)$$

or alternatively,

$$\frac{D\omega}{Dt} = \omega \cdot \nabla v + \nu \nabla^2 \omega. \quad (1.48)$$

It is also possible to obtain an energy equation by multiplying the Navier–Stokes equation by the velocity vector v . We employ subscripts here, noting that i and j can assume the values 1, 2, and 3, corresponding to the x, y , and z directions:

$$\rho v_j \frac{\partial}{\partial x_j} \left(\frac{1}{2} v_i v_i \right) = \frac{\partial}{\partial x_j} (\tau_{ij} v_i) - \tau_{ij} \frac{\partial v_i}{\partial x_j}. \quad (1.49)$$

$\tau_{i,j}$ is the symmetric stress tensor, and we are employing Stokes' simplification:

$$\tau_{ij} = -p \delta_{ij} + 2\mu S_{ij}. \quad (1.50)$$

δ is the Kronecker delta ($\delta_{ij} = 1$ if $i = j$, and zero otherwise) and S_{ij} is the strain rate tensor,

$$S_{ij} = \frac{1}{2} \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right]. \quad (1.51)$$

In the literature of fluid mechanics, the strain rate tensor is often written as it appears in eq. (1.51), but one may also find $S_{ij} = [\partial v_i / \partial x_j + \partial v_j / \partial x_i]$. Symmetric second-order tensors have three invariants (by invariant, we mean there is no change resulting from rotation of the coordinate system):

$$I_1(A) = \operatorname{tr}(A), \quad (1.52)$$

$$I_2(A) = \frac{1}{2} \left[(\operatorname{tr}(A))^2 - \operatorname{tr}(A^2) \right] \quad (1.53)$$

(which for a symmetric A is $I_2 = A_{11}A_{22} + A_{22}A_{33} + A_{11}A_{33} - A_{12}^2 - A_{23}^2 - A_{13}^2$), and

$$I_3(A) = \det(A). \quad (1.54)$$

The second invariant of the strain rate tensor is particularly useful to us; it is the double dot product of S_{ij} , which we write as $\sum_i \sum_j S_{ij} S_{ji}$. For rectangular coordinates, we obtain

$$I_2 = 2 \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right)^2 + \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right)^2. \quad (1.55)$$

You may recognize these terms; they are used to compute the production of thermal energy by viscous dissipation, and they can be very important in flow systems with large velocity gradients. We will see them again in Chapter 7.

We shall make extensive use of these relationships in this book. This is a good point to summarize the Navier–Stokes equations, so that we can refer to them as needed.

Rectangular coordinates

$$\begin{aligned} & \rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \\ &= -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x, \end{aligned} \quad (1.56a)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) \\ &= -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y, \end{aligned} \quad (1.56b)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) \\ &= -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z. \end{aligned} \quad (1.56c)$$

Cylindrical coordinates

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) \\ &= -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] \\ &+ \rho g_r, \end{aligned} \quad (1.57a)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) \\ &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] \\ &+ \rho g_\theta, \end{aligned} \quad (1.57b)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ &= -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z. \end{aligned} \quad (1.57c)$$

Spherical coordinates

$$\begin{aligned} & \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\ &= -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) \right. \\ &\quad \left. + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] + \rho g_r, \end{aligned} \quad (1.58a)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \right) \\ &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) \right. \\ &\quad \left. + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right] + \rho g_\theta \end{aligned} \quad (1.58b)$$

$$\begin{aligned} & \rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) \\ &= -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) \right. \\ &\quad \left. + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] + \rho g_\phi \end{aligned} \quad (1.58c)$$

These equations have attracted the attention of many eminent mathematicians and physicists; despite more than 160 years of very intense work, only a handful of solutions are known for the Navier–Stokes equation(s). White (1991) puts the number at 80, which is pitifully small compared to the number of flows we might wish to consider. The Clay Mathematics Institute has observed that “. . . although these equations were written down in the 19th century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier–Stokes equations.”

1.6 THE MEN FOR WHOM THE NAVIER–STOKES EQUATIONS ARE NAMED

The equations of fluid motion given immediately above are named after Claude Louis Marie Henri Navier (1785–1836) and Sir George Gabriel Stokes (1819–1903). There was no professional overlap between the two men as Navier died in 1836 when Stokes (a 17-year-old) was in his second year at Bristol College. Navier had been taught by Fourier at the Ecole Polytechnique and that clearly had a great influence upon his subsequent interest in mathematical analysis. But in the nineteenth century, Navier was known primarily as a bridge designer/builder who made important contributions to structural mechanics. His work in fluid mechanics was not as well known. Anderson (1997) observed that Navier *did not understand shear stress* and although he did not intend to derive the equations governing fluid motion *with* molecular friction, he did arrive at the proper form for those equations. Stokes himself displayed talent for mathematics while at Bristol. He entered Pembroke College at Cambridge in 1837 and was coached in mathematics by William Hopkins; later, Hopkins recommended hydrodynamics to Stokes as an

area ripe for investigation. Stokes set about to account for frictional effects occurring in flowing fluids and again the proper form of the equation(s) was discovered (but this time with intent). He became aware of Navier's work after completing his own derivation. In 1845, Stokes published "On the Theories of the Internal Friction of Fluids in Motion" recognizing that his development employed different assumptions from those of Navier. For a better glimpse into the personalities and lives of Navier and Stokes, see the biographical sketches written by O'Connor and Robertson²⁰⁰³ (*MacTutor History of Mathematics*). A much richer picture of Stokes the man can be obtained by reading his correspondence (especially between Stokes and Mary Susanna Robinson) in Larmor's memoir (1907).

1.7 SIR ISAAC NEWTON

Much of what we routinely use in the study of transport phenomena (and, indeed, in all of mathematics and mechanics) is due to Sir Isaac Newton. Newton, according to the contemporary calendar, was born on Christmas Day in 1642; by modern calendar, his date of birth was January 4, 1643. His father (also Isaac Newton) died prior to his son's birth and although the elder Newton was a wealthy landowner, he could neither read nor write. His mother, following the death of her second husband, intended for young Isaac to manage the family estate. However, this was a task for which Isaac had neither the temperament nor the interest. Fortunately, an uncle, William Ayscough, recognized that the lad's abilities were directed elsewhere and was instrumental in getting him entered at Trinity College Cambridge in 1661.

Many of Newton's most important contributions had their origins in the plague years of 1665–1667 when the University was closed. While home at Lincolnshire, he developed the foundation for what he called the "method of fluxions" (differential calculus) and he also perceived that integration was the inverse operation to differentiation. As an aside, we note that a fluxion, or differential coefficient, is the change in one variable brought about by the change in another, related variable. In 1669, Newton assumed the Lucasian chair at Cambridge (see the information compiled by Robert Bruen and also <http://www.lucasianchair.org/>) following Barrow's resignation. Newton lectured on optics in a course that began in January 1670 and in 1672 he published a paper on light and color in the *Philosophical Transactions of the Royal Society*. This work was criticized by Robert Hooke and that led to a scientific feud that did not come to an end until Hooke's death in 1703. Indeed, Newton's famous quote, "If I have seen further it is by standing on ye shoulders of giants," which has often been interpreted as a statement of humility appears to have actually been intended as an insult to Hooke (who was a short hunchback, becoming increasingly deformed with age).

Certainly Newton had a difficult personality with a dichotomous nature—he wanted recognition for his developments but was so averse to criticism that he was reticent about sharing his discoveries through publication. This characteristic contributed to the acrimony over who should be credited with the development of differential calculus, Newton or Leibniz. Indeed, this debate created a schism between British and continental mathematicians that lasted decades. But two points are absolutely clear: Newton's development of the "method of fluxions" predated Leibniz's work and each man used his own, unique, system of notation (suggesting that the efforts were completely independent). Since differential calculus ranks arguably as the most important intellectual accomplishment of the seventeenth century, one can at least comprehend the vitriol of this long-lasting debate. Newton used the Royal Society to "resolve" the question of priority; however, since he wrote the committee's report anonymously, there can be no claim to impartiality.

Newton also had a very contentious relationship with John Flamsteed, the first Astronomer Royal. Newton needed Flamsteed's lunar observations to correct the lunar theory he had presented in *Principia (Philosophiae Naturalis Principia Mathematica)*. Flamsteed was clearly reluctant to provide these data to Newton and in fact demanded Newton's promise not to share or further disseminate the results, a restriction that Newton could not tolerate. Newton made repeated efforts to obtain Flamsteed's observations both directly and through the influence of Prince George, but without success. Flamsteed prevailed; his data were not published until 1725, 6 years after his death.

There is no area in optics, mathematics, or mechanics that was not at least touched by Newton's genius. No less a mathematician than Lagrange stated that Newton's *Principia* was the greatest production of the human mind and this evaluation was echoed by Laplace, Gauss, and Biot, among others. Two anecdotes, though probably unnecessary, can be used to underscore Newton's preeminence: In 1696, Johann Bernoulli put forward the *brachistochrone* problem (*to determine the path in the vertical plane by which a weight would descend most rapidly from higher point A to lower point B*). Leibniz worked the problem in 6 months; Newton solved it *overnight* according to the biographer, John Conduitt, finishing at about 4 the next morning. Other solutions were eventually obtained from Leibniz, l'Hopital, and both Jacob and Johann Bernoulli. In a completely unrelated problem, Newton was able to determine the path of a ray by (effectively) solving a differential equation in 1694; Euler could not solve the same problem in 1754. Laplace was able to solve it, *but in 1782*.

It is, I suppose, curiously comforting to ordinary mortals to know that truly rare geniuses like Newton always seem to be flawed. His assistant Whiston observed that "Newton was of the most fearful, cautious and suspicious temper that I ever knew."

Furthermore, in the brief glimpse offered here, we have avoided describing Newton's interests in alchemy, history, and prophecy, some of which might charitably be characterized as peculiar. It is also true that work he performed as warden of the Royal Mint does not fit the reclusive scholar stereotype; as an example, Newton was instrumental in having the counterfeiter William Chaloner hanged, drawn, and quartered in 1699. Nevertheless, Newton's legacy in mathematical physics is absolutely unique. There is no other case in history where a single man did so much to advance the science of his era so far beyond the level of his contemporaries.

We are fortunate to have so much information available regarding Newton's life and work through both his own writing and exchanges of correspondence with others. A select number of valuable references used in the preparation of this account are provided immediately below.

The Correspondence of Isaac Newton, edited by H. W. Turnbull, FRS, University Press, Cambridge (1961).

The Newton Handbook, Derek Gjertsen, Routledge & Kegan Paul, London (1986).

Memoirs of Sir Isaac Newton, Sir David Brewster, reprinted from the Edinburgh Edition of 1855, Johnson Reprint Corporation, New York (1965).

A Short Account of the History of Mathematics, 6th edition, W. W. Rouse Ball, Macmillan, London (1915).

See also <http://www-groups.dcs.st-and.ac.uk> and <http://www.newton.cam.ac.uk>.

REFERENCES

- Anderson, J. D. *A History of Aerodynamics*, Cambridge University Press, New York (1997).
- Baker, G. L. and J. P. Gollub. *Chaotic Dynamics*, Cambridge University Press, Cambridge (1990).
- Baruh, H. Are Computers Hurting Education? *ASEE Prism*, p. 64 (October 2001).
- Batchelor, G. K. *An Introduction to Fluid Dynamics*, Cambridge University Press, Cambridge (1967).
- Bird, R. B., Stewart, W. E., and E. N. Lightfoot. *Transport Phenomena*, 2nd edition, Wiley, New York (2002).
- Bohren, C. F. Comment on "Newton's Law of Cooling—A Critical Assessment," by C. T. O'Sullivan. *American Journal of Physics*, 59:1044 (1991).
- Clay Mathematics Institute, www.claymath.org.
- Davis, H. T. *Introduction to Nonlinear Differential and Integral Equations*, Dover Publications, New York (1962).
- Fermi, E., Pasta, J., and S. Ulam. Studies of Nonlinear Problems, 1. *Report LA-1940* (1955).
- Landau, L. D. and E. M. Lifshitz. *Fluid Mechanics*, Pergamon Press, London (1959).
- Larmor, J., editor. *Memoir and Scientific Correspondence of the Late Sir George Gabriel Stokes*, Cambridge University Press, New York (1907).
- Lorenz, E. N. Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences*, 20:130 (1963).
- Milne-Thomson, L. M. *Jacobian Elliptic Function Tables: A Guide to Practical Computation with Elliptic Functions and Integrals*, Dover, New York (1950).
- O'Connor, J. J. and E. F. Robertson. MacTutor History of Mathematics, www.history.mcs.st-andrews.ac.uk (2003).
- Packard, N. H., Crutchfield, J. P., Farmer, J. D., and R. S. Shaw. Geometry from a Time Series. *Physical Review Letters*, 45:712 (1980).
- Porter, M. A., Zabusky, N. J., Hu, B., and D. K. Campbell. Fermi, Pasta, Ulam and the Birth of Experimental Mathematics. *American Scientist*, 97:214 (2009).
- Powers, D. L. *Boundary Value Problems*, 2nd edition, Academic Press, New York (1979).
- Rossler, O. E. An Equation for Continuous Chaos. *Physics Letters*, 57A:397 (1976).
- Schlichting, H. *Boundary-Layer Theory*, 6th edition, McGraw-Hill, New York (1968).
- Stokes, G. G. On the Theories of the Internal Friction of Fluids in Motion. *Transactions of the Cambridge Philosophical Society*, 8:287 (1845).
- Truesdell, C. The Present Status of the Controversy Regarding the Bulk Viscosity of Liquids. *Proceedings of the Royal Society of London*, A226:1 (1954).
- Vaughn, M. T. *Introduction to Mathematical Physics*, Wiley-VCH, Weinheim (2007).
- White, F. M. *Viscous Fluid Flow*, 2nd edition, McGraw-Hill, New York (1991).