Chapter 7

POWERS, EXPONENTS, AND ROOTS

Chapter Check-In

- ❑ Operations with powers and exponents
- ❑ Square roots and cube roots
- ❑ Simplifying and approximating square roots

Powers and Exponents

Before you begin working with powers and exponents, some basic definitions are necessary.

Exponents

An **exponent** is a positive or negative number placed above and to the right of a quantity. It expresses the power to which the quantity is to be raised or lowered. In 4^3, 3 is the exponent. It shows that 4 is to be used as a factor three times: $4 \times 4 \times 4$ (multiplied by itself twice). 4^3 is read as four to the *third power* (or *four cubed*).

$$2^4 = 2 \times 2 \times 2 \times 2 = 16$$
$$3^2 = 3 \times 3 = 9$$
$$5^3 = 5 \times 5 \times 5 = 125$$

Remember: $x^1 = x$ and $x^0 = 1$ when x is any number (other than 0).

$2^1 = 2$	$2^0 = 1$
$3^1 = 3$	$3^0 = 1$
$4^1 = 4$	$4^0 = 1$

Negative exponents

If the exponent is negative, such as 4^{-2}, the number and exponent may be dropped under the number 1 in a fraction to remove the negative sign.

Example 1: Simply the following by removing the exponents.

(a) 4^{-2} (b) 5^{-3} (c) 2^{-4} (d) 3^{-1}

(a) $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

(b) $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$

(c) $2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

(d) $3^{-1} = \frac{1}{3^1} = \frac{1}{3}$

Squares and cubes

Two specific types of powers should be noted: **squares** and **cubes.** To square a number, just multiply it by itself (the exponent is 2). For example, 6 squared (written 6^2) is 6×6, or 36. 36 is called a **perfect square** (the square of a whole number). Following is a partial list of perfect squares:

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$8^2 = 64$
$9^2 = 81$
$10^2 = 100$
$11^2 = 121$
$12^2 = 144$

To **cube** a number, just multiply it by itself twice (the exponent is 3). For example, 5 cubed (written 5^3 is $5 \times 5 \times 5$, or 125. 125 is called a **perfect cube** (the cube of a whole number). Following is a partial list of perfect cubes.

$1^3 = 1 \times 1 \times 1 = 1$
$2^3 = 2 \times 2 \times 2 = 8$
$3^3 = 3 \times 3 \times 3 = 27$

$$4^3 = 4 \times 4 \times 4 = 64$$
$$5^3 = 5 \times 5 \times 5 = 125$$
$$6^3 = 6 \times 6 \times 6 = 216$$
$$7^3 = 7 \times 7 \times 7 = 343$$

Operations with powers and exponents

To multiply two numbers with exponents, if the base numbers are the same, simply keep the base number and add the exponents.

Example 2: Multiply the following, leaving the answers with exponents.

(a) $2^3 \times 2^5$ (b) $3^2 \times 3^5$ (c) $5^4 \times 5^7$

(a) $2^3 \times 2^5 = 2^{(3+5)} = 2^8$

$(2 \times 2 \times 2) \times (2 \times 2 \times 2 \times 2 \times 2) = 2^8$

(b) $3^2 \times 3^5 = 3^{(2+5)} = 3^7$

(c) $5^4 \times 5^7 = 5^{(4+7)} = 5^{11}$

To divide two numbers with exponents, if the base numbers are the same, simply keep the base number and subtract the second exponent from the first, or the exponent of the denominator from the exponent of the numerator.

Example 3: Divide the following, leaving the answers with exponents.

(a) $5^6 \div 5^2$ (b) $\frac{8^7}{8^3}$

(a) $5^6 \div 5^2 = 5^{(6-2)} = 5^4$

(b) $\frac{8^7}{8^3} = 8^{(7-3)} = 8^4$

To multiply or divide numbers with exponents, if the base numbers are different, you must simplify each number with an exponent first and then perform the operation.

Example 4: Simplify and perform the operation indicated.

(a) $2^3 \times 3^2$ (b) $6^2 \div 2^3$

(a) $2^3 \times 3^2 = 8 \times 9 = 72$

(b) $6^2 \div 2^3 = 36 \div 8 = 4\frac{4}{8} = 4\frac{1}{2}$

For problems such as those in Example 4, some shortcuts are possible.

To add or subtract numbers with exponents, whether the base numbers are the same or different, you must simplify each number with an exponent first and then perform the indicated operation.

Example 5: Simplify and perform the operation indicated.

(a) $3^2 - 2^3$ (b) $5^2 + 3^3$ (c) $4^2 + 9^3$ (d) $2^3 - 2^2$

(a) $3^2 - 2^3 = 9 - 8 = 1$

(b) $5^2 + 3^3 = 25 + 27 = 52$

(c) $4^2 + 9^3 = 16 + 729 = 745$

(d) $2^3 - 2^2 = 8 - 4 = 4$

If a number with an exponent is taken to another power $(4^2)^3$, simply keep the original base number and multiply the exponents.

Example 6: Multiply the following and leave the answers with exponents.

(a) $(6^3)^2$ (b) $(3^2)^4$ (c) $(5^4)^3$

(a) $(6^3)^2 = 6^{(3\times2)} = 6^6$

(b) $(3^2)^4 = 3^{(2\times4)} = 3^8$

(c) $(5^4)^3 = 5^{(4\times3)} = 5^{12}$

Square Roots and Cube Roots

Note: Square and cube roots and operations with them are often included in algebra books.

Square roots

To find the **square root** of a number, you want to find some number that when multiplied by itself gives you the original number. In other words, to find the square root of 25, you want to find the number that when multiplied by itself gives you 25. The square root of 25, then, is 5. The symbol for the square root is $\sqrt{\ }$. Following is a partial list of perfect (whole number) square roots.

$\sqrt{0} = 0$ $\sqrt{16} = 4$ $\sqrt{64} = 8$

$\sqrt{1} = 1$ $\sqrt{25} = 5$ $\sqrt{81} = 9$

$\sqrt{4} = 2$ $\sqrt{36} = 6$ $\sqrt{100} = 10$

$\sqrt{9} = 3$ $\sqrt{49} = 7$

Note: If no sign (or a positive sign) is placed in front of the square root, the positive answer is required. No sign means that a positive is understood. Only if a negative sign is in front of the square root is the negative answer required. This notation is used in many texts, as well as this book. Therefore,

$$\sqrt{4} = 2 \text{ and } -\sqrt{4} = -2$$

Cube roots

To find the **cube root** of a number, you want to find some number that when multiplied by itself twice gives you the original number. In other words, to find the cube root of 8, you want to find the number that when multiplied by itself twice gives you 8. The cube root of 8, then, is 2 because $2 \times 2 \times 2 = 8$. Notice that the symbol for cube root is the radical sign with a small three (called the *index*) above and to the left $\sqrt[3]{\ }$. Other roots are similarly defined and identified by the index given. (In square root, an index of 2 is understood and usually not written.) Following is a partial list of *perfect* (whole number) cube roots.

$$\sqrt[3]{0} = 0 \qquad \sqrt[3]{27} = 3$$
$$\sqrt[3]{1} = 1 \qquad \sqrt[3]{64} = 4$$
$$\sqrt[3]{8} = 2 \qquad \sqrt[3]{125} = 5$$

Approximating square roots

To find the square root of a number that is not a perfect square, it is necessary to find an approximate answer by using the procedure given in Example 7.

Example 7: Approximate $\sqrt{42}$.

$$\sqrt{42} \text{ is between } \sqrt{36} \text{ and } \sqrt{49}$$

$$\sqrt{36} < \sqrt{42} < \sqrt{49}$$

$$\sqrt{36} = 6$$

$$\sqrt{49} = 7$$

Therefore, $6 < \sqrt{42} < 7$

Since 42 is almost halfway between 36 and 49, $\sqrt{42}$ is almost halfway between $\sqrt{36}$ and $\sqrt{49}$. So $\sqrt{42}$ is approximately 6.5. To check, multiply the following:

$$6.5 \times 6.5 = 42.25$$

or about 42.

Example 8: Approximate $\sqrt{71}$.

$$\sqrt{64} < \sqrt{71} < \sqrt{81}$$
$$8 < \sqrt{71} < 9$$

Since $\sqrt{71}$ is slightly closer to $\sqrt{64}$ than it is to $\sqrt{81}$,

$$8 < 8.4 < 9$$
$$\sqrt{71} \approx 8.4$$

Check the answer.

$$\begin{array}{r} 8.4 \\ \times\, 8.4 \\ \hline 336 \\ 672 \\ \hline 70.56 \approx 71 \end{array}$$

Example 9: Approximate $\sqrt{\frac{300}{15}}$.

First, perform the operation under the radical.

$$\sqrt{\frac{300}{15}} = \sqrt{20}$$
$$\sqrt{16} < \sqrt{20} < \sqrt{25}$$
$$4 < \sqrt{20} < 5$$

Since $\sqrt{20}$ is slightly closer to $\sqrt{16}$ than it is to $\sqrt{25}$,

$$4 < 4.4 < 5$$
$$\sqrt{\frac{300}{15}} \approx 4.4$$

Square roots of nonperfect squares can be approximated, looked up in tables, or found by using a calculator. You may want to keep these two in mind, because they are commonly used.

$$\sqrt{2} \approx 1.414 \quad \sqrt{3} \approx 1.732$$

Simplifying square roots

Sometimes, you have to simplify square roots or write them in simplest form. In fractions, $\frac{2}{4}$ can be reduced to $\frac{1}{2}$. In square roots, $\sqrt{32}$ can be simplified to $4\sqrt{2}$. To simplify a square root, first factor the number under the $\sqrt{\ }$ into two factors, one of which is the largest possible perfect square. (Perfect square numbers are 1, 4, 9, 25, 49, and so on.)

Example 10: Simplify $\sqrt{32}$.

$$\sqrt{32} = \sqrt{16 \times 2}$$

Then take the square root of the perfect square number.

$$\sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4 \times \sqrt{2}$$

Finally, write it as a single expression

$$4\sqrt{2}$$

Example 11: Simplify $\sqrt{24}$.

$$\begin{aligned} \sqrt{24} &= \sqrt{4 \times 6} \\ &= \sqrt{4} \times \sqrt{6} \\ &= 2 \times \sqrt{6} \\ &= 2\sqrt{6} \end{aligned}$$

To check, square the number on the outside of the radical and multiply it by the number on the inside.

$$\begin{aligned} 2^2 \times 6 &= 4 \times 6 \\ &= 24 \end{aligned}$$

Example 12: Simplify $\sqrt{75}$.

$$\begin{aligned} \sqrt{75} &= \sqrt{25 \times 3} \\ &= \sqrt{25} \times \sqrt{3} \\ &= 5 \times \sqrt{3} \\ &= 5\sqrt{3} \end{aligned}$$

Remember: Most square roots cannot be simplified, because they are already in simplest form, such as $\sqrt{7}$, $\sqrt{10}$, and $\sqrt{15}$.

Chapter Checkout

Q&A

1. $5^4 =$ _____.

2. $6^{-2} =$ _____.

3. $3^2 \times 3^5 =$ _____. (with exponents)

4. $5^8 \div 5^3 =$ _____. (with exponents)

5. $4^2 - 3^3 =$ _____.

6. $(4^3)^2 =$ _____. (with exponents)

7. $\sqrt[3]{125} =$ _____.

8. Approximate $\sqrt{29}$ to the nearest tenth.

9. Simplify $\sqrt{72}$.

Answers: 1. 625 **2.** $1/36$ **3.** 3^7 **4.** 5^5 **5.** –11 **6.** 4^6 **7.** 5 **8.** 5.4 **9.** $6\sqrt{2}$