Chapter 1
PRECALCULUS PREREQUISITES

Chapter Check-In

- ❑ Defining common number groups
- ❑ Writing inequalities as intervals
- ❑ Understanding algebraic properties
- ❑ Working with exponents, radicals, polynomials, and rational expressions
- ❑ Finding solutions to equations and inequalities
- ❑ Constructing linear equations

A strong algebraic background is essential to success in precalculus. Before you can begin exploring its more advanced topics, you must first have a firm grip on the fundamentals. In this chapter, you'll review and practice foundational concepts and skills.

Classifying Numbers

Many times throughout your precalculus course, you'll be manipulating specific kinds of numbers, so it's important to understand how mathematicians classify numbers and what kinds of major classifications exist. Be aware that numbers can fall into more than one group. Just as an American citizen can also be classified as a North American citizen or a citizen of Earth, numbers may also belong to numerous categories simultaneously.

The following groups, or sets, of numbers are generally agreed on by mathematicians as the most common classifications of numbers. They are listed here in order of size, from smallest to largest:

- **Natural numbers.** The set of numbers you've used since you were very young when counting (as such, the *natural numbers* can also be called the *counting numbers*): {1, 2, 3, 4, 5, 6, ...}.

- **Whole numbers.** The *whole numbers* include all of the *natural numbers* and, also, the number 0: {0, 1, 2, 3, 4, 5, ...}.
- **Integers.** All of the whole numbers and their opposites make up the set of *integers*. In other words, any number without an extra decimal or fraction attached to it is considered an integer: {..., –3, –2, –1, 0, 1, 2, 3, ...}. Because *integers* contain no obvious fractions or decimals, some students are tempted to refer to them as *whole numbers*, but that is not completely accurate, because the set of *whole numbers* does not include negative numbers.
- **Rational numbers.** A number is classified as *rational* if one of the following conditions hold true.

  The number can be expressed as a fraction. (In other words, the number can be rewritten as $\frac{a}{b}$, where a and b are integers, and $b \neq 0$.)

  The number is a terminating decimal, in other words a decimal that ends (such as 6.25) rather than continues on infinitely.

  The number is a decimal that repeats in an infinite pattern (such as 5.297297297297...).

  Basically, any number that can be written as a fraction is *rational.*

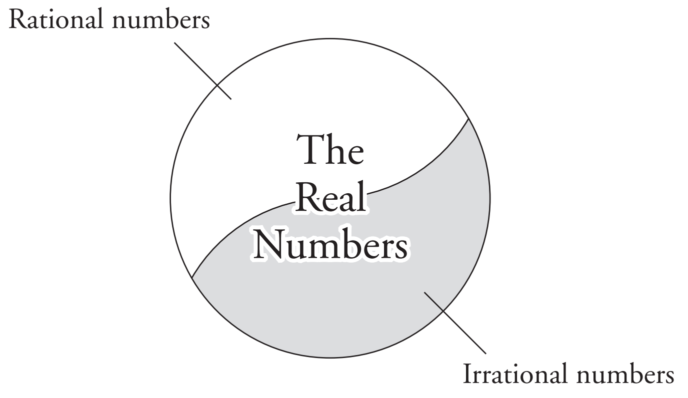
Example 1: Show that any integer must also be a rational number.

Any integer a can also be rewritten as $\frac{a}{1}$, since dividing by 1 will not alter the value of the integer. Because a can be expressed as a fraction whose numerator and denominator are both integers, a must be rational by definition.

- **Irrational numbers.** A number that cannot be written as a fraction is considered *irrational*. The most obvious indicator of an irrational number is a decimal that doesn't infinitely repeat itself yet never terminates. For example, π is an irrational number whose decimal equivalent 3.14159265359... never ends and never follows any obvious repeating pattern. Many radicals, like $\sqrt{3}$, are irrational numbers.
- **Real numbers.** The *real numbers* are made up of the *rational numbers* and the *irrational numbers* grouped together, as shown in Figure 1-1.
- **Complex numbers.** *Complex numbers* differ from the *real numbers* in appearance quite starkly. Complex numbers usually have two distinct parts and look like $a + bi$, where a is the *real part*, bi is the *imaginary part*, where i is equal to the imaginary value $\sqrt{-1}$. However,

numbers need not contain both parts to be considered complex. In fact, any real number is automatically complex. For example, since the real number 3 can be written as $3 + 0i$, 3 is a complex number. It just contains no imaginary part.

Figure 1-1 The rational and the irrational numbers together comprise the entire set of real numbers. Note that this drawing is not to scale. Far more irrational than rational numbers exist.



Interval Notation

Traditional inequality statements can be rewritten using **interval notation**, a shorthand method that expresses the same meaning but usually in a more compact and intuitive manner. This is largely due to the fact that interval notation clearly defines the boundaries of the inequality with which you're working.

Bounded intervals

If you're given an inequality that is bounded on both sides by a real number, that statement can be rewritten as a **bounded interval**. To create a bounded interval, write the two numerical endpoints of the interval in order, always from lowest to highest. (The interval will almost look like a coordinate pair.) Then, indicate whether that point should be included on the interval. If it should, use a bracket with that endpoint; if it should not, use a parenthesis.

Example 2: Rewrite the following inequality statements using interval notation.

(a) $-5 \le x \le 3$

Because the inequality signs stipulate less than *or equal to*, you must include the endpoints in the interval. Had equality not been a possibility, you would not include those endpoints. Use brackets to indicate inclusion: [–5,3].

(b) $1 > x > 0$

Even though this interval is written so that the upper boundary is on the left, interval notation still requires you to write them in order from lesser to greater. Use parentheses to indicate that the endpoints are not included in the interval: (0,1).

(c) $-2 \le x < 4$

The lower endpoint is included while the upper is not: [–2,4).

If both endpoints of the interval are included (as in part [a] of Example 2), the interval is said to be **closed**. On the other hand, if neither endpoint is included (as in part [b] of Example 3), it is an **open interval**.

Unbounded intervals

Sometimes, only one endpoint of an interval is explicitly defined and the other is implied. For instance, consider the inequality $x > 3$. Clearly, the lower boundary of the interval is 3, but what is the upper boundary? Because there is no finite value given for the upper endpoint, you use infinity. If one or more of the endpoints of an interval are understood to be infinite, the interval is said to be **unbounded**.

You will use two different infinite boundaries:

- ∞, if the boundary is infinitely positive (it is used as the upper bound of the interval)
- $-\infty$, if the boundary is infinitely negative (it is used as the lower bound of the interval)

Infinity is technically not a real number, which means you can never use a bracket to indicate its inclusion in the interval. Instead, always use a parenthesis.

Example 3: Rewrite the following inequality statements using interval notation:

(a) $x > -1$

The lower bound of the interval is –1, and the upper bound is infinitely large, since any positive number will make this inequality statement true. The lower boundary should not be included, because the relationship is "greater than," not "greater than or equal to": $(-1,\infty)$.

(b) $x \leq 3$

In this interval, 3 is the upper boundary. If the lower boundary of an interval is infinite, you must indicate this by using negative infinity: $(-\infty,3]$.

(c) All real numbers

Any real number, from the infinitely negative to the infinitely positive, should be included in this interval: $(-\infty,\infty)$.

Algebraic Properties

Properties, also called *laws* or *axioms*, are foundational mathematical principles that are assumed true. Although there is no way to irrefutably prove properties, they make enough inherent common sense to be universally agreed on by mathematicians. It's a good thing they are, because these laws form the backbone of algebra.

The associative property

Given a string of numbers added together, you may group the numbers in any order you wish and it will not affect the answer you get. This is the basic premise of the **associative property** for addition. In other words, no matter what numbers are *associated* together, you will get the same result in the end.

$$(1 + 3) + 5 = 1 + (3 + 5)$$
$$4 + 5 = 1 + 8$$
$$9 = 9$$

The associative property also holds true for multiplication, but it fails for both subtraction and division. Here are the official mathematical definitions for its two incarnations:

- **The associative property for addition:**

  $(a + b) + c = a + (b + c)$

- **The associative property for multiplication:**

  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Note that the symbol · is used here to indicate multiplication rather than the other traditional symbol for multiplication, ×. This is because it's easy to confuse the operation × with the variable *x* when you're working a problem.

The commutative property

This property (like its sister, the associative property) works only for addition and multiplication. In essence, it says that given a string of numbers being added or a string of numbers being multiplied, the order in which you complete that operation doesn't matter.

$$3 \cdot 9 = 9 \cdot 3$$
$$27 = 27$$

- **The commutative property for addition:**

  $a + b = b + a$

- **The commutative property for multiplication:**

  $a \cdot b = b \cdot a$

The distributive property

According to the **distributive property**, if terms are being added or subtracted within parentheses and a number appears "outside" that group of terms, you can multiply that outer number through to every number within those parentheses.

$$a(b + c) = ab + ac$$

Example 4: Rewrite using the distributive property:

$$3(x - 7)$$

Multiply every term in the parentheses by 3:

$$3 \cdot x - 3 \cdot 7$$
$$3x - 21$$

Identity elements

Fixed numbers called **identity elements** exist for both the operations of addition and multiplication. These elements do not alter a number's value (or *identity*) when the operation is applied to them. The *identity element for addition* (also called the *additive identity*) is 0, because if you add 0 to any number, you get back what you started with:

$$2 + 0 = 2$$

Similarly, the *identity element for multiplication* (also called the *multiplicative identity*) is 1, since multiplying any number by 1 will not change that number's value:

$$3 \cdot 1 = 3$$

These identity elements are important because they are a major component in the *inverse properties.*

Inverse properties

Once again, the operations of addition and multiplication have a property specific to them. In both cases, an **inverse property** assures you that no matter the input, there is a way to "cancel it out."

- **Additive inverse property:** For any real number a, there exists a real number $-a$ (the *opposite* of a) so that $a + (-a) = 0$:

$$4 + (-4) = 0$$

- **Multiplicative inverse property:** For any non-zero real number a, there exists a real number $\frac{1}{a}$ so that $a \cdot \frac{1}{a} = 1$:

$$7 \cdot \frac{1}{7} = 1$$

Note that when you "undo" addition and multiplication using these inverse properties, the result will be the *identity element* for the corresponding operation.

Exponential Expressions

Repeated multiplication can be rewritten using **exponents**, small numbers written above and to the right of the **base** number, both to clarify and simplify your notation. Rather than write "$x \cdot x \cdot x$," you can write "x^3," which is read "x to the third **power**." The *power* of an exponent is the number of times the object is multiplied by itself. Therefore, in the expression x^3, x is considered the *base* and 3 is the *power*.

There are six important rules you should know when undertaking any arithmetic involving exponents:

- **Rule 1:** $x^a \cdot x^b = x^{a+b}$

  If two exponential expressions with identical bases are multiplied, the result is that base raised to an exponent equal to the sum of the two powers:

$$x^4 \cdot x^7 = x^{4+7} = x^{11}$$

- **Rule 2:** $\frac{x^a}{x^b} = x^{a-b}$

  If two exponential expressions with identical bases are divided, the result is that base raised to an exponent equal to the power in the numerator minus the power in the denominator:

$$\frac{x^8}{x^5} = x^{8-5} = x^3$$

- **Rule 3:** $\left(x^a\right)^b = x^{a \cdot b}$

  If an exponential expression is itself raised to a power, the result is the base raised to the product of the two powers:

$$\left(x^2\right)^6 = x^{2 \cdot 6} = x^{12}$$

- **Rule 4:** $\left(x^a y^b\right)^c = x^{ac} y^{bc}$

  If numerous exponential factors are raised to a power, multiply the outer power times each of the inner powers.

$$\left(x^2 y^5\right)^3 = x^{2 \cdot 3} y^{5 \cdot 3} = x^6 y^{15}$$

- **Rule 5:** $x^{-a} = \frac{1}{x^a}$ and $\frac{1}{x^{-b}} = x^b$

  A negative exponent indicates that the expression is in the wrong part of the fraction. To make the exponent positive again (no algebraic expression is completely simplified until it contains no negative exponents), move the exponential expression to the other side of the fraction bar. For instance, if it is in the numerator, move it to the denominator, and leave the base alone.

$$\frac{x^{-2}}{y^{-3}} = \frac{y^3}{x^2}$$

- **Rule 6:** $x^0 = 1$ (if $x \neq 0$)

  Any real number raised to the 0 power is equal to 1 (with the exception of 0^0, which does not have a real number value).

Example 5: Simplify using the exponential rules:

(a) $\dfrac{x^3y^5z^2}{xy^7z^2}$

Rewrite the fraction using Rule 2. Since the x in the denominator has no visible exponent, it is understood to be 1.

$$x^{3-1}y^{5-7}z^{2-2}$$

$$x^2y^{-2}z^0$$

A completely simplified solution does not contain negative exponents. Apply Rule 5 to achieve that goal. In addition, rewrite z^0 as 1.

$$\frac{x^2}{y^2}$$

(b) $(x^2y^3)(x^7yz^3)$

You can rearrange the terms thanks to the commutative propery and then add exponential powers of like bases, thanks to Rule 1. Again, since the y in the second group of parentheses has no exponent visible, it is understood to be 1.

$$x^{2+7}y^{3+1}z^3$$

$$x^9y^4z^3$$

(c) $\left(\dfrac{x^2y^{-2}}{z^{-3}}\right)^{-2}$

Begin by applying Rule 4:

$$\frac{x^{-4}y^4}{z^6}$$

Use Rule 5 to eliminate negative exponents:

$$\frac{y^4}{x^4z^6}$$

Radical Expressions

Although most of the time the exponents you'll see will be integers, you may run across some fractional powers as well. These types of powers translate into **radicals** (also called *roots*):

$$x^{a/b} = \sqrt[b]{x^a} \text{ or } \left(\sqrt[b]{x}\right)^a$$

You can use either notation to rewrite the fractional power as a radical. In some cases, one form will be more useful than the other when you are simplifying.

A typical radical, $\sqrt[n]{x^a}$ contains two parts: the **index** (the small number in front of the radical) and the **radicand**, the quantity within the radical symbol. It is read "The *n*th root of *x* to the *a*th power." Note that if no *index* is given for the radical, the *index* is understood to be 2.

Some students find radicals easier to understand if they think of the notation as a question. For example, the radical $\sqrt[3]{8}$ asks the question "What number multiplied by itself 3 times is equal to 8?" The answer is 2, so $\sqrt[3]{8} = 2$.

Properties of radicals

Because radicals are really exponents in disguise (even if they are fractional exponents), radicals possess the same properties as exponents. In addition, radicals have these properties:

- $\sqrt[n]{x^a y^b} = \sqrt[n]{x^a} \cdot \sqrt[n]{y^b}$

  Factors multiplied together inside of a radical can be broken up and written as the product of two radicals with the same index as the original. That is to say, the root of a product is equal to the product of the individual roots:

- $\sqrt[n]{\dfrac{x^a}{y^b}} = \dfrac{\sqrt[n]{x^a}}{\sqrt[n]{y^b}}$

  Just like multiplication, division problems surrounded by radicals can be broken up into separate, smaller radicals as well. So, the root of the quotient is equal to the quotient of the individual roots.

Simplifying radicals

The most common task you'll face in the study of radicals is the need to simplify radical expressions.

Example 6: Use the properties of radicals to simplify these expressions:

(a) $\sqrt{200x^2y}$

Your goal will be to break this radical into two different radicals, one containing all **perfect squares** and the other containing everything else. *Perfect squares* are quantities generated by multiplying some value by itself.

$$\sqrt{100 \cdot 2 \cdot x^2 \cdot y}$$
$$\sqrt{100x^2} \cdot \sqrt{2y}$$

Both 100 and x^2 are perfect squares (since 100 = 10 · 10 and $x^2 = x \cdot x$); the leftmost radical will be eliminated.

$$10|x|\sqrt{2y}$$

You might not have expected the absolute value signs. They are rare but necessary when you have this situation: $\sqrt[n]{x^n}$ and n is an even integer. This precaution ensures that the answer is positive, because a radical with an even index must always be positive.

(b) $\sqrt[3]{-108x^2y^8}$

Again, split up the radical, but this time put all of the **perfect cubes** (values generated by multiplying the same thing by itself three times) together:

$$\sqrt[3]{-27y^6} \cdot \sqrt[3]{4x^2y^2}$$
$$\left(-3y^2\right)\sqrt[3]{4x^2y^2}$$

There is no need to worry about absolute value signs because the index of this radical is odd.

Operations with radicals

It is a bit more complicated to add and subtract radical expressions than it is to multiply and divide them. In fact, radicals must have the same index and radicand in order to perform addition and subtraction, but that is not the case for multiplication and division.

Example 7: Simplify the following expressions:

(a) $5\sqrt{2} - 3\sqrt{8}$

While the indices are the same (they are both 2), the radicands appear different at first glance. That changes when you simplify the expression:

$$5\sqrt{2} - 3\sqrt{4}\sqrt{2}$$
$$5\sqrt{2} - 6\sqrt{2}$$

Now that they share the same radicand as well, you can combine the coefficients and get $-\sqrt{2}$.

(b) $\left(\sqrt{x}\right)\left(\sqrt[3]{x^2}\right)$

Begin by rewriting the radicals as exponential expressions:

$$x^{1/2} \cdot x^{2/3}$$

Apply Rule 1 for exponential expressions:

$$x^{1/2 + 2/3} = x^{3/6 + 4/6} = x^{7/6} = \sqrt[6]{x^7}$$

You may write your final answer in either exponential or radical form; they are equivalent.

Rationalizing expressions

Some teachers require that you **rationalize** your answers, when appropriate. This means they don't want an answer containing a radical sign in its denominator.

Example 8: Rationalize the following fraction:

$$\frac{x}{\sqrt{3}}$$

To eliminate the radical, multiply both the numerator and denominator by a value of $\sqrt{3}$. This is the equivalent of multiplying by 1, so it doesn't change the value of the fraction, and it creates a perfect square in the denominator.

$$\frac{x}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{x\sqrt{3}}{\sqrt{9}} = \frac{x\sqrt{3}}{3}$$

Polynomial Expressions

Polynomials are strings of **terms** added to or subtracted from one another. Each term is made up of numbers and variables (usually raised to whole number powers) multiplied together. For example, the polynomial

$$4x^3 - 2x^2 + x + 7$$

is made up of four terms. The **coefficient** is the numerical value preceding the variable in each term. The first term, $4x^3$, has a *coefficient* of 4, and the second term has a coefficient of –2. The **degree** of this polynomial, defined by the highest exponent found in the polynomial, is 3. The **leading coefficient** is the coefficient accompanying the variable raised to the highest exponential value. In this example, the leading coefficient is 4.

Classifying polynomials

Polynomials are typically categorized either according to the number of terms they possess or according to the degree of the polynomial.

- **Classifying according to number of terms**

  A polynomial containing only one term is called a **monomial**. If two terms are present, the polynomial is considered a **binomial**, while three terms indicates a **trinomial**. No commonly used terms are available that indicate a polynomial containing four, five, or more terms.

- **Classifying according to degree**

  It's easy to categorize a polynomial according to its degree. Simply look for the highest exponent within the polynomial. Table 1-1 gives the classifications based on a polynomial's degree.

Table 1-1 Degree Classifications for Polynomials

Degree	*Category*	*Example*
0	constant	7
1	linear	$-x + 7$
2	quadratic	$5x^2 + x + 7$
3	cubic	$x^3 - 1$
4	quartic	$-7x^4 - x^3 + 2x^2 + 5x - 3$
5	quintic	$x^5 - x^2 + x$

The classifications in Table 1-1 are not the only ones; additional names exist for polynomials of higher degree, but these are the most commonly used.

Adding and subtracting polynomials

Remember, you can only add or subtract radicals that contain the exact same radicand and index. Similarly, you can only add or subtract polynomial terms that contain the same variables and exponents. Such terms are called **like terms**.

Example 9: Simplify the following expressions:

(a) $(x^3 - 6x^2 + 3x + 4) + (4x^3 + 2x^2 - 10x - 5)$

According to the commutative and associative properties of addition, you can reorder the terms and group them differently. Rewrite them as groups of like terms:

$$(x^3 + 4x^3) + (-6x^2 + 2x^2) + (3x - 10x) + (4 - 5)$$
$$5x^3 - 4x^2 - 7x - 1$$

(b) $(x^2 + 2x + 1) - 2(x + 6)$

Use the distributive property to simplify before combining like terms:

$$(x^2 + 2x + 1) - 2x - 12$$
$$x^2 + (2x - 2x) + (1 - 12)$$
$$x^2 - 11$$

Multiplying polynomials

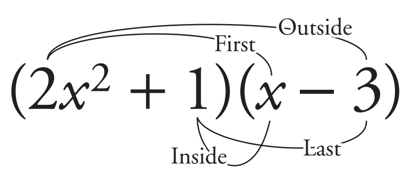
Terms need not be like terms in order to multiply them together. In fact, to multiply polynomials together, all you need is the distributive property.

Example 10: Multiply the following expressions and simplify your answer:

(a) $(2x^2 + 1)(x - 3)$

You may use the **FOIL method** to find the product. *FOIL* is a mnemonic device meaning "First, Outside, Inside, Last," describing which terms must be multiplied together. Figure 1-2 explains what is meant by each of the four groups in *FOIL*.

Figure 1-2 Each letter in FOIL stands for a pair of terms that must be multiplied together.



Multiply the first terms together, then the outer, the inner, and the last terms, and add the products where possible.

$$(2x^2 \cdot x) + (2x^2 \cdot -3) + (1 \cdot x) + (1 \cdot -3)$$
$$2x^3 - 6x^2 + x - 3$$

(b) $(x - 2)(x^2 - 4x + 5)$

In order for you to use the FOIL method, both polynomials must be binomials, and that's not the case here. When FOIL fails, simply distribute each term in the first polynomial into the second polynomial:

$$x(x^2 - 4x + 5) - 2(x^2 - 4x + 5)$$
$$x^3 - 4x^2 + 5x - 2x^2 + 8x - 10$$
$$x^3 - 6x^2 + 13x - 10$$

You will review how to divide polynomials in Chapter 3.

Rational Expressions

Just as any number that can be expressed as a fraction is called a *rational number*, any expression written as a fraction is called a **rational expression**. Operations on rational expressions follow the same governing rules as operations on fractions.

Adding and subtracting rational expressions

All fractions must have common denominators before they can be combined via addition or subtraction.

Example 11: Simplify the expression:

$$\frac{3}{x} + \frac{x-1}{x+2} - \frac{x}{x-5}$$

The **least common denominator** (LCD) for this expression is $x(x + 2)(x - 5)$, since that is the smallest expression containing one of each of the pieces of every denominator. Multiply each fraction by the values necessary to get that denominator. Remember, multiplying by a fraction which has the same numerator and denominator is the same as multiplying by 1, so you're not changing the values of the original fractions.

$$\frac{3}{x} \cdot \frac{(x+2)(x-5)}{(x+2)(x-5)} + \frac{x-1}{x+2} \cdot \frac{x(x-5)}{x(x-5)} - \frac{x}{x-5} \cdot \frac{x(x+2)}{x(x+2)}$$

$$\frac{3\left(x^2 - 3x - 10\right)}{x(x+2)(x-5)} + \frac{(x-1)\left(x^2 - 5x\right)}{x(x+2)(x-5)} - \frac{x^2(x+2)}{x(x+2)(x-5)}$$

Now, all the numerators can be written over the LCD.

$$\frac{\left(3x^2 - 9x - 30\right) + \left(x^3 - 6x^2 + 5x\right) - \left(x^3 + 2x^2\right)}{x\,(x+2)\,(x-5)}$$

$$\frac{-5x^2 - 4x - 30}{x^3 - 3x^2 - 10x}$$

Multiplying rational expressions

You do not need common denominators in order to multiply rational expressions. To find the product of two fractions, simply multiply the numerator of the first times the numerator of the second; then do likewise with the denominators.

Example 12: Simplify the expression:

$$\left(\frac{x+3}{5}\right)\left(\frac{x^2-1}{2x}\right)$$

$$\frac{(x+3)\left(x^2-1\right)}{5 \cdot 2x}$$

$$\frac{x^3 + 3x^2 - x - 3}{10x}$$

Simplifying complex fractions

When fractions are divided, the result is a **complex fraction**, a fraction that itself contains fractions. To simplify such fractions, you will employ a method that changes the division problem into multiplication.

Example 13: Simplify the complex fraction:

$$\frac{\dfrac{x+1}{x}}{\dfrac{x^2}{2x-3}}$$

Begin by rewriting the complex fraction as a division problem:

$$\frac{x+1}{x} \div \frac{x^2}{2x-3}$$

Take the reciprocal of the second fraction (turn it upside down) and change the division sign to multiplication:

$$\frac{x+1}{x} \cdot \frac{2x-3}{x^2}$$

Multiply as you would ordinary fractions:

$$\frac{2x^2 - x - 3}{x^3}$$

Equations and Inequalities

Your primary task as a precalculus student will be to solve equations and inequalities using a variety of techniques, so it's worthwhile to make sure you have a good mastery of the techniques you should know thus far.

Solving equations

To solve an equation for a variable (that is, to isolate the variable on one side of the equal sign), you may do any of the following:

- Add or subtract the same quantity on both sides of the equal sign.
- Multiply or divide both sides of the equal sign by the same non-zero quantity. However, make sure that you do not multiply or divide by a variable quantity, if at all possible. Doing so could result in additional or lost solutions, respectively.
- Cross-multiply to eliminate fractions.

$$\frac{a}{b} = \frac{c}{d} \text{ becomes } ad = bc$$

Example 14: Solve the following equations:

(a) $2x - 4 = 3(x - 9) + 2$

Distribute the 3, and then move the variables to the left and the constants to the right sides of the equation.

$$2x - 4 = 3x - 27 + 2$$
$$-x = -21$$

Divide both sides by –1 to get the answer: $x = 21$.

(b) $\frac{x+2}{x-1} + \frac{x}{x+2} = 2$

Subtract the second fraction from both sides:

$$\frac{x+2}{x-1} = 2\left(\frac{x+2}{x+2}\right) - \frac{x}{x+2}$$
$$\frac{x+2}{x-1} = \frac{x+4}{x+2}$$

Cross-multiply and solve:

$$(x+2)(x+2) = (x-1)(x+4)$$

$$x^2 + 4x + 4 = x^2 + 3x - 4$$

$$x = -8$$

(c) $|3x+1| = 7$

If only the quantity in absolute values appears on the left side of the equation, you can rewrite it as two different equations, both without absolute value signs. In one, you simply set the sides equal; in the other, the right-hand side of the equation is written as its opposite:

$$3x+1=7 \text{ or } 3x+1=-7$$

$$3x=6 \text{ or } 3x=-8$$

$$x=2 \text{ or } -\frac{8}{3}$$

Solving linear inequalities

Linear inequalities are treated almost exactly like equations. The only difference is that multiplying or dividing both sides of the inequality by a negative value reverses the inequality sign. For example, ≥ becomes ≤ and < becomes >.

Example 15: Give the solutions in interval notation:

(a) $3x + 4 < 5x + 7(x - 1)$

Distribute the 7 and isolate the *x* terms as if this were an equation:

$$3x + 4 < 5x + 7x - 7$$

$$3x + 4 < 12x - 7$$

$$-9x < -11$$

To finish, you have to divide by –9, so reverse the inequality symbol:

$$x > \frac{11}{9}$$

In interval form, the answer is $\left(\frac{11}{9}, \infty\right)$

(b) $-5 \le 2x + 3 < 13$

Subtract 3 from all parts of the inequality, and then divide everything by 2 to isolate the *x*:

$$-8 \le 2x < 10$$

$$-4 \le x < 5$$

The answer, in interval form, is [–4,5).

Solving absolute value inequalities

Just as absolute value equations require you to solve two equations, absolute value inequalities require you to solve two inequalities. The procedures are different for problems involving less-than signs and those involving greater-than signs.

Example 16: Give the solutions in interval notation:

(a) $|x-4|-3<6$

Isolate the absolute value quantity on the left side:

$$|x-4|<9$$

Transform this into a double inequality, removing the absolute value signs. The quantity on the far left will be the opposite of the quantity on the far right, and the inequality signs match:

$$-9<x-4<9$$

Solve this just like Example 15(b):

$$-5<x<13$$

The answer is (–5,13).

(b) $|3x+1|\ge 4$

Inequalities involving the greater-than symbol must be rewritten as two linear inequalities. In the first, simply drop the absolute value signs. In the second, reverse the sign and change the constant on the right to its opposite:

$$3x+1\ge 4 \quad \text{or} \quad 3x+1\le -4$$

$$3x\ge 3 \quad \text{or} \quad 3x\le -5$$

$$x\ge 1 \quad \text{or} \quad x\le -\frac{5}{3}$$

In interval form, the answer is $[1,\infty)$ or $(-\infty,-\frac{5}{3}]$. You can replace the word "or" with the symbol $\cup$; both notations are correct.

Special inequality cases

Whenever you are presented with inequality problems containing rational expressions or polynomials of a degree higher than one (such as quadratics or cubics), you must use an altogether different method. Here are the steps you should follow.

1. Move all terms to the left side of the inequality, leaving only 0 on the right side.
2. Find the **critical numbers**, the values for which the left side of the inequality either equals 0 or is undefined. (Remember, a fraction equals 0 when its numerator equals 0 and is undefined when its denominator equals 0.)
3. Draw a number line and mark the critical points on it. Use a closed dot to represent included points (points that could be a solution) and an open dot to represent unattainable points (such as places where the expression is undefined or where the inequality sign does permit the possibility of equality).
4. Treat those dots as boundaries that split the number line into intervals and choose one value (called a **test point**) from each segment, *between the critical numbers*.
5. Each interval whose test point makes the original inequality true is a solution.

Example 17: Give the solutions in interval notation:

(a) $\frac{x+1}{x-2} \geq 3$

Subtract 3 from both sides and simplify:

$$\frac{x+1}{x-2} - 3 \cdot \frac{x-2}{x-2} \geq 0$$

$$\frac{x+1-3x+6}{x-2} \geq 0$$

$$\frac{-2x+7}{x-2} \geq 0$$

The numerator equals 0 when $x = \frac{7}{2}$, and the denominator equals 0 when $x = 2$; both are critical numbers. Since the inequality includes the possibility of equality, you use a solid dot for $\frac{7}{2}$. However, 2 makes the fraction undefined and cannot be a solution; use an open dot for it, as shown in Figure 1-3.

Figure 1-3 The critical numbers 2 and $\frac{7}{2}$ break the number line into three distinct intervals.

The numbers $x = 0$, 3, and 5 belong to the pictured intervals, from left to right. When you plug each into the original inequality, only $x = 3$ makes it true. Therefore, the interval in which it belongs is the solution: $(2, \frac{7}{2}]$.

(b) $2x^2 - 5x - 3 < 0$

Wherever the trinomial equals 0, place an open dot critical point on the number line. You need to factor in order to find these values. (A brief review of factoring is given in Chapter 3.)

$$(x - 3)(2x + 1)$$

Set both factors equal to 0 to get critical numbers of $x = 3$ and $-\frac{1}{2}$. Choose test values from the resulting intervals of $(-\infty, -\frac{1}{2})$, $(-\frac{1}{2}, 3)$ and $(3, \infty)$, and test them in the original inequality. The correct answer is $(-\frac{1}{2}, 3)$.

Finding Linear Equations

You need only two items to write the equation of any line: its **slope** (a fraction describing how quickly the line rises vertically compared to how it rises horizontally) and any point on the line. Once you have that information, plug it into the correct spots of **point-slope form** for a linear equation:

$$y - y_1 = m(x - x_1)$$

where m is the slope and the point you were given on the line is (x_1, y_1).

If you do not know the slope of the line but are given two points, (x_1, y_1) and (x_2, y_2), you can calculate the slope using this equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 18: Find the equations of the following lines:

(a) line l, which has slope –2 and passes through point (–1,5)

Set $m = -2$, $x_1 = -1$, and $y_1 = 5$; plug these values into point-slope form:

$$y - y_1 = m(x - x_1)$$
$$y - 5 = -2(x - (-1))$$
$$y - 5 = -2x - 2$$

If you solve this equation for y, you get the **slope-intercept form** for a line ($y = mx + b$), where m is once again the slope and b is the line's y-intercept:

$$y = -2x + 3$$

(b) line k, which passes through points (–2,6) and (3,–5)

Begin by calculating the slope:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-5 - 6}{3 - (-2)} \\ &= -\frac{11}{5} \end{aligned}$$

Now use point-slope form with either of the given points:

$$y - y_1 = m(x - x_1)$$
$$y - 6 = -\frac{11}{5}\left(x - (-2)\right)$$
$$y - 6 = -\frac{11}{5}x - \frac{22}{5}$$
$$y = -\frac{11}{5}x + \frac{8}{5}$$

No matter which point you choose when plugging into point-slope form, you'll get the identical answer when you solve for y and express your answer in slope-intercept form.

(c) line n, which has y-intercept –3 and is parallel to $y = 2x + 1$

Lines that are parallel have equal slopes, so the slope of line n will be 2. (Perpendicular lines have slopes that are negative reciprocals.) Because you already know the y-intercept for line n, use slope-intercept form: $y = 2x - 3$.

Chapter Checkout

Q&A

1. True or False: All rational numbers are also real numbers.
2. Express the inequality $x \geq 7$ in interval notation.
3. True or False: (1 + 2) + 3 = 3 + (1 + 2) because of the associative property of addition.
4. Simplify this radical: $\sqrt{72x^3y}$.
5. Express the solution in interval notation: $|x - 2| + 3 > 5$.
6. Express the solution in interval notation: $x^2 \leq 9$.
7. Find the equation of the line through points (0,–3) and (–2,7), and write the linear equation in slope-intercept form.

Answers: 1. T **2.** $[7,\infty)$ **3.** F **4.** $6|x|\sqrt{2xy}$ **5.** $(-\infty,0)\cup(4,\infty)$ **6.** $[-3,3]$ **7.** $y = -5x - 3$