Chapter 3

Flow Processes and Hydrostatic Forces

3.1 CONTROL VOLUME APPROACH FOR HYDROSYSTEMS

Hydrosystem processes transform the space and time distribution of water in hydrologic systems throughout the hydrologic cycle, in natural and human-made hydraulic systems, and in water resources systems that include both hydrologic and hydraulic systems. The commonality of all hydrosystems is the physical laws that define the flow of fluid in these systems. A consistent mechanism for developing these physical laws is called the *control volume approach*.

The simplified concept of a system is very important in the control volume approach because of the extreme complexity of hydrosystems. Typically a system is defined from the fluids viewpoint is defined as a given quantity of mass. A *system* is also a set of connected parts that form a whole. For the present discussion the fluids viewpoint will be used, in which the system has a *system boundary* or *control surface* (CS) as shown in Figure 3.1.1. A control surface is the surface that surrounds the control volume. The control surface can coincide with physical boundaries such as the wall of a pipe or the boundary of a watershed. Part of the control surface may be a hypothetical surface through which fluid flows.

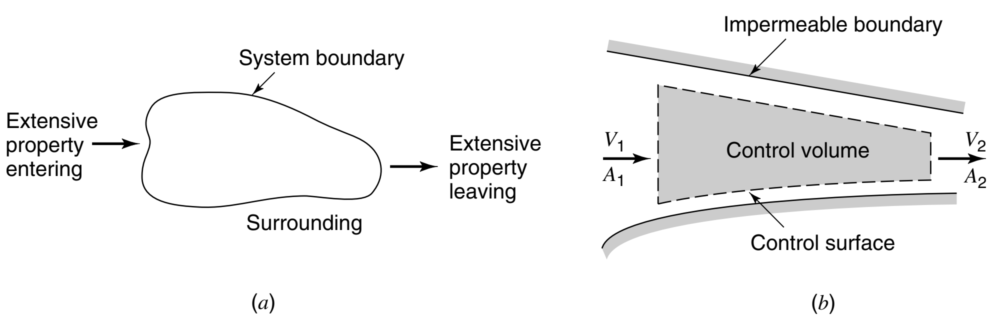

Figure 3.1.1 Control volume approach. (*a*) System and surrounding; (*b*) Control volume as a system.

Two properties, *extensive properties* and *intensive properties*, are used in the control volume approach to apply physical properties for discrete masses to a fluid flowing continuously through a control volume. Extensive properties are related to the total mass of the system (control volume), whereas intensive properties are independent of the amount of fluid. The extensive properties are mass m, momentum mV, and energy E. Corresponding intensive properties are mass per unit mass, momentum per unit mass, which is velocity v, and energy per unit mass e. In other words, for an extensive property B, the corresponding intensive property β is defined as the quantity of B per unit mass, $\beta = dB/dm$. Both the extensive and intensive properties can be scalar or vector quantities.

The relationship between intensive and extensive properties for a given system is defined by the following integral over the system:

$$B = \int_{\text{system}} \beta\, dm = \int \beta \rho\, d\forall \tag{3.1.1}$$

where dm and $d\forall$ are the differential mass and differential volume, respectively, and ρ is the fluid density.

The volume rate of flow past a given area A is expressed as

$$Q = \mathbf{V} \cdot \mathbf{A} \tag{3.1.2}$$

where $\mathbf{V}$ is the velocity vector of flow and $\mathbf{A}$ is the area vector, which is directed normal to the area and points outward from the control volume.

For the control volume in Figure 3.1.1, the net flowrate $\dot{Q}$ is

$$\begin{aligned} \dot{Q} &= Q_{\text{out}} - Q_{\text{in}} \\ &= \mathbf{V}_2 \cdot \mathbf{A}_2 - \mathbf{V}_1 \cdot \mathbf{A}_1 \\ &= \sum_{\text{CS}} \mathbf{V} \cdot \mathbf{A} \end{aligned} \tag{3.1.3}$$

In other words, the dot product $\mathbf{V} \cdot \mathbf{A}$ for all flows in and out of a control volume is the net rate of outflow.

The mass rate of flow out of the control volume is

$$\frac{dm}{dt} = \dot{m} = \sum_{\text{CS}} \rho \mathbf{V} \cdot \mathbf{A} \tag{3.1.4}$$

The rate of flow of extensive property B is the product of the mass rate and the intensive property:

$$\frac{dB}{dt} = \dot{B} = \sum_{\text{CS}} \beta \rho \mathbf{V} \cdot \mathbf{A} \tag{3.1.5}$$

If the velocity varies across the flow section, then it must be integrated across the section, so that the above equation for the rate of flow of extensive property $\dot{B}$ from the control volume becomes

$$\dot{B} = \int_{\text{CS}} \beta \rho \mathbf{V} \cdot d\mathbf{A} \tag{3.1.6}$$

Considering the system in Figure 3.1.2, the control volume is defined by the control surface at time t(I + II) with extensive property B_t. At time $t + \Delta t$ the control volume, defined by the control surface, (II + III) has moved and has extensive property $B_{t+\Delta t}$. The rate of change of extensive property B is

$$\frac{dB}{dt} = \lim_{\Delta t \to 0} \left[\frac{B_{t+\Delta t} - B_t}{\Delta t} \right] \tag{3.1.7}$$

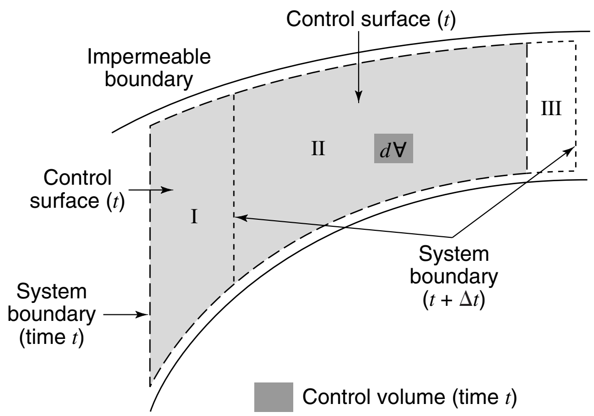

Figure 3.1.2 Control volume at times t and $t + \Delta t$.

The mass of the system at time $t + \Delta t$, $m_{\text{sys},t+\Delta t}$, is

$$m_{\text{sys},t+\Delta t} = m_{t+\Delta t} + \Delta m_{\text{out}} - \Delta m_{\text{in}} \tag{3.1.8}$$

where $m_{t+\Delta t}$ = mass of fluid within the control volume at time $t + \Delta t$
Δm_{out} = mass of fluid that has moved out of the control volume in time Δt
Δm_{in} = mass of fluid that has moved into the control volume in time Δt

The extensive property of the system at time $t + \Delta t$ is

$$B_{\text{sys}} = B_{\text{CV},t+\Delta t} + \Delta B_{\text{out}} - \Delta B_{\text{in}} \tag{3.1.9}$$

where $B_{\text{CV},t+\Delta t}$ = amount of extensive property in the control volume at time $t + \Delta t$
ΔB_{out} = amount of extensive property of the system that has moved out of the control volume in time Δt
ΔB_{in} = amount of extensive property of the system that has moved into the control volume in time Δt

The time rate of change of extensive property of the system is

$$\frac{dB_{\text{sys}}}{dt} = \lim_{\Delta t \to 0}\left[\frac{\left(B_{\text{CV},t+\Delta t} + \Delta B_{\text{out}} - \Delta B_{\text{in}}\right) - B_{\text{CV},t}}{\Delta t}\right] \tag{3.1.10}$$

The expression can be rearranged to yield

$$\begin{aligned}\frac{dB_{\text{sys}}}{dt} &= \lim_{\Delta t \to 0}\left[\frac{B_{\text{CV},t+\Delta t} - B_{\text{CV},t}}{\Delta t}\right] + \lim_{\Delta t \to 0}\left[\frac{\Delta B_{\text{out}} - \Delta B_{\text{in}}}{\Delta t}\right] \\ &= \left\{\begin{matrix}\text{Rate of change with} \\ \text{respect to time of} \\ \text{extensive property} \\ \text{in the control volume}\end{matrix}\right\} + \left\{\begin{matrix}\text{Net flow of} \\ \text{extensive property} \\ \text{from the control} \\ \text{volume}\end{matrix}\right\} \\ &= \frac{dB_{\text{CV}}}{dt} + \frac{dB}{dt}\end{aligned} \tag{3.1.11}$$

The derivative $\frac{dB_{\text{CV}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \beta\rho \, d\forall$ and $\frac{dB}{dt}$ is defined by equation (3.1.5), so that the *control volume equation for one-dimensional flow* becomes

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{\text{CV}} \beta\rho \, d\forall + \sum_{\text{CS}} \beta\rho \mathbf{V} \cdot \mathbf{A} \tag{3.1.12}$$

The above equation for the general control volume equation was derived for one-dimensional flow so that the rate of flow of B at each section is $\beta\rho\mathbf{V}\cdot\mathbf{A}$. A more general form for rate of flow of an extensive property considers the velocity as variable across a section. Using equation (3.1.6), then, the *general control volume equation* is expressed as

$$\frac{dB_{\text{sys}}}{dt}=\frac{d}{dt}\int_{CV}\beta\rho\,d\forall+\int_{CS}\beta\rho\,\mathbf{V}\cdot d\mathbf{A} \tag{3.1.13}$$

This general control volume equation (also referred to as the *Reynolds transport theorem*) states that the total rate of change of extensive property of a flow is equal to the rate of change of extensive property stored in the control volume, $\frac{d}{dt}\int_{CV}\beta\rho\,d\forall$, plus the net rate of outflow of extensive property through the control surface, $\int_{CS}\beta\rho\,\mathbf{V}\cdot d\mathbf{A}$.

Throughout this book the general control volume equation (approach) is applied to develop continuity, energy, and momentum equations for hydrosystem (hydrologic and hydraulic) processes.

3.2 CONTINUITY

In order to write the continuity equation, the extensive property is mass ($B = m$) and the intensive property $\beta = dB/dm = 1$. By the law of conservation of mass, the mass of a system is constant, therefore $dB/dt = dm/dt = 0$. The general form of the continuity equation is then

$$0=\frac{d}{dt}\int_{CV}\rho\,d\forall+\int_{CS}\rho\,\mathbf{V}\cdot d\mathbf{A} \tag{3.2.1}$$

which is the *integral equation of continuity for an unsteady, variable-density flow*. Equation (3.2.1) can be rewritten as

$$\int_{CS}\rho\,\mathbf{V}\cdot d\mathbf{A}=-\frac{d}{dt}\int_{CV}\rho\,d\forall \tag{3.2.2}$$

which states that the net rate of outflow of mass from the control volume is equal to the rate of decrease of mass within the control volume.

For flow with constant density, equation (3.2.2) can be expressed as

$$\int_{CS}\mathbf{V}\cdot d\mathbf{A}=-\frac{d}{dt}\int_{CV}d\forall \tag{3.2.3}$$

The continuity equation for flow with a uniform velocity across the flow section and constant density is expressed as

$$\sum_{CS}\mathbf{V}\cdot\mathbf{A}=-\frac{d}{dt}\int_{CV}d\forall \tag{3.2.4}$$

For a *constant-density*, *steady one-dimensional flow,* such as water flowing in a conduit, the velocity is the mean velocity, then

$$\sum_{CS}\mathbf{V}\cdot\mathbf{A}=0 \tag{3.2.5}$$

For pipe conduit flow we consider a control volume between two locations of the pipe, at sections 1 and 2, then the continuity equation is

$$-V_1A_1+V_2A_2=0 \tag{3.2.6a}$$

or

$$V_1 A_1 = V_2 A_2 \tag{3.2.6b}$$

or

$$Q_1 = Q_2 \tag{3.2.6c}$$

For a *constant-density unsteady flow*, consider the integral $\int_{CV} d\forall$ as the volume of fluid stored in a control volume denoted by S, so that

$$\frac{d}{dt}\int_{CV} d\forall = \frac{dS}{dt} \tag{3.2.7}$$

The net outflow is defined as

$$\begin{aligned}\int_{CS} \mathbf{V}\cdot d\mathbf{A} &= \int_{\text{outlet}} \mathbf{V}\cdot d\mathbf{A} + \int_{\text{inlet}} \mathbf{V}\cdot d\mathbf{A} \\ &= Q(t) - I(t)\end{aligned} \tag{3.2.8}$$

Then the integral equation of continuity is determined by substituting equations (3.2.7) and (3.2.8) into equation (3.2.2) to obtain

$$Q(t) - I(t) = -\frac{dS}{dt} \tag{3.2.9}$$

or more commonly expressed as

$$\frac{dS}{dt} = I(t) - Q(t) \tag{3.2.10}$$

This continuity expression is used extensively in describing hydrologic processes.

EXAMPLE 3.2.1

A river section is defined by two bridges. At a particular time the flow at the upstream bridge is 100 m^3/s and at the same time the flow at the downstream bridge is 75 m^3/s. At this particular time, what is the rate at which water is being stored in the river section, assuming no losses?

SOLUTION

Using the continuity equation (3.2.10) yields

$$\begin{aligned}\frac{dS}{dt} &= Q_{\text{up}}(t) - Q_{\text{down}}(t) \\ &= 100\ \text{m}^3/\text{s} - 75\ \text{m}^3/\text{s} \\ &= 25\ \text{m}^3/\text{s}\end{aligned}$$

EXAMPLE 3.2.2

A reservoir has the following monthly inflows and outflows in relative units:

Month	*J*	*F*	*M*	*A*
Inflows	10	5	0	5
Outflows	5	5	10	0

If the reservoir contains 30 units of water in storage at the beginning of the year, how many units of water in storage are there at the end of April?

SOLUTION

The integral equation (3.2.10) of continuity is used to perform a routing of flows into and out of the reservoir. Because the inflow and outflows are for discrete time intervals, the continuity equation (3.2.10) can be reformulated as

$$dS = I(t)dt - Q(t)dt$$

and integrated over time intervals $j = 1, 2, \ldots J$ of each length Δt:

$$\int_{S_{j-1}}^{S_j} dS = \int_{(j-1)\Delta t}^{j\Delta t} I(t)dt - \int_{(j-1)\Delta t}^{j\Delta t} Q(t)dt$$

or

$$S_j - S_{j-1} = I_j - Q_j \text{ for } j = 1, 2, \ldots$$

$$\Delta S_j = I_j - Q_j$$

where I_j and Q_j are the volumes of inflow and outflow for the jth time interval. The cumulative storage is $S_{j+1} = S_j + \Delta S_j$. For the first interval of time,

$$\Delta S_1 = I_1 - Q_1 = 10 - 5 = 5$$

Then $S_2 = S_1 + \Delta S_1 = 30 + 5 = 35$. The remaining computations are:

Time	I_j	Q_j	ΔS_j	S_j
1	10	5	5	30
2	5	5	0	35
3	0	10	−10	25
4	5	0	5	30

EXAMPLE 3.2.3

Consider the steady flow of water through a nozzle in which the upstream diameter of $D_2 = 30$ cm reduces to a downstream diameter of $D_2 = 20$ cm. For a flowrate of 0.08 m^3/s, compute the mean velocities for the upstream and downstream diameters.

SOLUTION

Using the continuity equations (3.2.6), $Q = V_1A_1 = V_2A_2$, we get

$$V_1 = Q/A_1 = \frac{0.08}{\left[\pi (30/100)^2/4\right]} = 1.13 \text{ m/s}$$

$$V_2 = Q/A_2 = \frac{0.08}{\left[\pi (20/100)^2/4\right]} = 2.55 \text{ m/s}$$

3.3 ENERGY

This section uses the first law of thermodynamics along with the control volume approach to develop the energy equation for fluid flow in hydrologic and hydraulic processes. An energy balance for hydrologic and hydraulic processes considers an accounting of all inputs and outputs of energy to and from a system. By the *first law of thermodynamics*, the rate of change of energy, E, with time is the rate at which heat is transferred into the fluid, dH/dt, minus the rate at which the fluid does work on the surroundings, dW/dt, expressed as

$$\frac{dE}{dt} = \frac{dH}{dt} - \frac{dW}{dt} \tag{3.3.1}$$

The total energy of a fluid system is the sum of the internal energy E_u the kinetic energy E_k, and the potential energy E_p; thus

$$E = E_u + E_k + E_p \tag{3.3.2}$$

The extensive property is the amount of energy in the system, $B = E$:

$$B = E_u + E_k + E_p \tag{3.3.3}$$

and the intensive property is

$$\beta = \frac{dB}{dm} = e = e_u + e_k + e_p \tag{3.3.4}$$

where e represents the energy per unit mass. Also, the rate of change of extensive property with respect to time is

$$\frac{dB}{dt} = \frac{dE}{dt} = \frac{dH}{dt} - \frac{dW}{dt} \tag{3.3.5}$$

The *energy balance equation* is now derived by substituting β (equation (3.3.4)) and dB/dt (equation (3.3.5)) into the general control volume equation (3.1.12),

$$\frac{dE}{dt} = \frac{dH}{dt} - \frac{dW}{dt} = \frac{d}{dt}\int_{CV} e\rho\, d\forall + \sum_{CS} e\rho \mathbf{V}\cdot\mathbf{A} \tag{3.3.6}$$

Next we can replace e by equation (3.3.4):

$$\frac{dH}{dt} - \frac{dW}{dt} = \frac{d}{dt}\int_{CV}\left(e_u + e_k + e_p\right)\rho\, d\forall + \sum_{CS}\left(e_u + e_k + e_p\right)\rho\, \mathbf{V}\cdot\mathbf{A} \tag{3.3.7}$$

The kinetic energy per unit mass e_k is the total kinetic energy of mass with velocity V divided by the mass m:

$$e_k = \frac{mV^2/2}{m} = \frac{V^2}{2} \tag{3.3.8}$$

The potential energy per unit mass e_p is the weight of the fluid $\gamma\forall$ times the centroid elevation z of the mass divided by the mass:

$$e_p = \frac{\gamma\,\forall z}{m} = \frac{\gamma\,\forall z}{\rho\forall} = gz \tag{3.3.9}$$

because $\gamma/\rho = g$.

Now the *general energy equation for unsteady variable density flow* can be written as

$$\frac{dH}{dt} - \frac{dW}{dt} = \frac{d}{dt}\int_{CV}\left(e_u + \frac{1}{2}V^2 + gz\right)\rho\, d\forall + \sum_{CS}\left(e_u + \frac{1}{2}V^2 + gz\right)\rho \mathbf{V}\cdot\mathbf{A} \tag{3.3.10}$$

For steady flow, equation (3.3.10) reduces to

$$\frac{dH}{dt} - \frac{dW}{dt} = \sum_{CS}\left(e_u + \frac{1}{2}V^2 + gz\right)\rho \mathbf{V}\cdot\mathbf{A} \tag{3.3.11}$$

The work done by a system on its surroundings can be divided into *shaft work*, W_s, and *flow work*, W_f. Flow work is the result of pressure force as the system moves through space and shaft work is any other work besides the flow work. In the control volume in Figure 3.1.2, the force on the upstream end of the fluid is p_1A_1 and the distance traveled over time Δt is $l_1 = V_1\Delta t$. Work done on the surrounding fluid as a result of this force is then the product of the force p_1A_1 in the direction of motion and the distance traveled, $V_1\Delta t$. The work force on the upstream end is then

$$W_{f_1} = -V_1 p_1 A_1 \Delta t \tag{3.3.12a}$$

and on the downstream end is

$$W_{f_2} = V_2 p_2 A_2 \Delta t \tag{3.3.12b}$$

At the upstream end a negative sign must be used because the pressure force on the surrounding fluid acts in the opposite direction to the motion of the system boundary. The rate of work at the upstream and downstream ends are respectively

$$\frac{dW_{f_1}}{dt} = -V_1 p_1 A_1 \tag{3.3.13}$$

and

$$\frac{dW_{f_2}}{dt} = V_2 p_2 A_2 \tag{3.3.14}$$

The rate of flow work can then be expressed in general terms as

$$\frac{dW_f}{dt} = p\,\mathbf{V}\cdot\mathbf{A} \tag{3.3.15}$$

or for all streams passing through the control volume as

$$\frac{dW_f}{dt} = \sum_{\text{CS}} p\mathbf{V}\cdot\mathbf{A} = \sum_{\text{CS}} \frac{p}{\rho}\,\rho\mathbf{V}\cdot\mathbf{A} \tag{3.3.16}$$

The net rate of work on the system can now be expressed as

$$\frac{dW}{dt} = \frac{dW_s}{dt} + \sum_{\text{CS}} \frac{p}{\rho}\,\rho\,\mathbf{V}\cdot\mathbf{A} \tag{3.3.17}$$

Using equation (3.3.17), the *general energy equation* (3.3.10) *for unsteady variable density flow* can be expressed as

$$\frac{dH}{dt} - \frac{dW_s}{dt} - \sum_{\text{CS}} \frac{p}{\rho}\,\rho\,\mathbf{V}\cdot\mathbf{A} = \frac{d}{dt}\int_{\text{CV}} \left(e_u + \frac{1}{2}V^2 + gz\right)\rho\, d\forall + \sum_{\text{CS}} \left(e_u + \frac{1}{2}V^2 + gz\right)\rho\,\mathbf{V}\cdot\mathbf{A} \tag{3.3.18}$$

which can be written as

$$\frac{dH}{dt} - \frac{dW_s}{dt} = \frac{d}{dt}\int_{\text{CV}} \left(e_u + \frac{1}{2}V^2 + gz\right)\rho\, d\forall + \sum_{\text{CS}} \left(\frac{p}{\rho} + e_u + \frac{1}{2}V^2 + gz\right)\rho\,\mathbf{V}\cdot\mathbf{A} \tag{3.3.19}$$

For steady flow, equation (3.3.19) reduces to

$$\frac{dH}{dt} - \frac{dW_s}{dt} = \sum_{\text{CS}} \left(\frac{p}{\rho} + e_u + \frac{1}{2}V^2 + gz\right)\rho\,\mathbf{V}\cdot\mathbf{A} \tag{3.3.20}$$

EXAMPLE 3.3.1

Determine an expression based upon the energy concept that relates the pressures at the upstream and downstream ends of the nozzle in example 3.2.3, assuming steady flow, neglecting change in internal energy, and assuming $dH/dt = 0$ and $dW_s/dt = 0$.

SOLUTION

Using the energy equation (3.3.20) for steady flow yields

$$\frac{dH}{dt} - \frac{dW_s}{dt} = \sum_{\text{CS}} \left(\frac{p}{\rho} + e_u + \frac{1}{2}V^2 + gz\right)\rho\,\mathbf{V}\cdot\mathbf{A}$$

Neglecting dH/dt and dW_s/dt the above energy equation can be expressed as

$$\int_{A_2}\left(\frac{p_2}{\rho}+e_{u_2}+\frac{1}{2}V_2^2+gz_2\right)\rho\, \mathrm{V}_2\, dA_2-\int_{A_1}\left(\frac{p_1}{\rho}+e_{u_1}+\frac{1}{2}V_1^2+gz_1\right)\rho\, \mathrm{V}_1\, dA_1=0$$

which can be modified to

$$\int_{A_2}\left(\frac{p_2}{\rho}+e_{u_2}+gz_2\right)\rho\, \mathrm{V}_2\, dA_2+\int_{A_2}\frac{\rho\, V_2^3}{2}dA_2-\int_{A_1}\left(\frac{p_1}{\rho}+e_{u_1}+gz_1\right)\rho\, \mathrm{V}_1\, dA_1-\int_{A_1}\frac{\rho\, V_1^3}{2}dA_1=0$$

For hydrostatic conditions, $\left(\frac{p}{\rho}+e_u+gz\right)$ is constant across the system, which allows these term to be taken outside the integral:

$$\left(\frac{p_2}{\rho}+e_{u_2}+gz_2\right)\int_{A_2}\rho V_2 dA_2+\int_{A_2}\frac{\rho\, V_2^3}{2}dA_2-\left(\frac{p_1}{\rho}+e_{u_1}+gz_1\right)\int_{A_1}\rho V_1 dA_1-\int_{A_1}\frac{\rho\, V_1^3}{2}dA_1=0$$

The term $\int \rho V dA$ is the mass rate of flow, $\dot{m}$, and the term $\int \frac{\rho\, V^3}{2}\mathrm{d}A=\dot{m}\frac{V^2}{2}$, so

$$\left(\frac{p_2}{\rho}+e_{u_2}+gz_2\right)\dot{m}\;+\;\dot{m}\frac{V_2^2}{2}-\left(\frac{p_1}{\rho}+e_{u_1}+gz_1\right)\dot{m}-\dot{m}\frac{V_1^2}{2}=0$$

Dividing through by $\dot{m}g$ and rearranging yields

$$\frac{p_1}{\rho\, g}+\frac{e_{u_1}}{g}+z_1+\frac{V_1^2}{2g}=\frac{p_2}{\rho\, g}+\frac{e_{u_2}}{g}+z_2+\frac{V_2^2}{2g}$$

$\gamma = \rho g$ and rearranging yields

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2+\frac{e_{u_2}-e_{u_1}}{g}$$

Neglecting changes in internal energy, $(e_{u_2} - e_{u_1})/g = 0$

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

Assuming the control volume is horizontal, $z_1 = z_2$, then

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}$$

This energy equation relates the pressures assuming steady flow, $z_1 = z_2$, neglecting change of internal energy in the fluid and assuming $dH/dt = 0$ and $dW_s/dt = 0$.

EXAMPLE 3.3.2

For the nozzle in example 3.2.3, determine the pressure change through the nozzle between the upstream and downstream end of the nozzle. Assume steady flow, neglect changes in internal energy of the fluid, assume $dH/dt = 0$ and $dW_s/dt = 0$, and say that the nozzle is horizontal. Assume the temperature is 20°C.

SOLUTION For example 3.2.3, the velocities determined were $V_1 = 1.13$ m/s and $V_2 = 2.55$ m/s. Using the energy equation derived in example 3.3.1 yields

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

$$p_1 - p_2 = \left(V_2^2 - V_1^2\right)\frac{\gamma}{2g}$$

$$= \left[(2.55)^2 - (1.13)^2\right] \times \frac{9.79 \text{ kN/m}^3}{2 \times 9.81 \text{ m/s}^2}$$

$$= (5.226 \text{ m}^2/\text{s}^2)\ (0.499 \text{ kN s}^2/\text{m}^4)$$

$$= 2.608 \text{ kN/m}^2 = 2.608 \text{ kPa} = 2608 \text{ Pa}$$

The pressure change is a pressure decrease of 2608 Pa.

3.4 MOMENTUM

In order to derive the general momentum equation for fluid flow in a hydrologic or hydraulic system, we use the control volume approach along with Newton's second law. *Newton's second law* states that the summation of all external forces on a system is equal to the rate of change of momentum of the system

$$\sum \mathbf{F} = \frac{d(\text{momentum})}{dt} \tag{3.4.1}$$

To apply the control volume approach the extensive property is momentum, $B = m\mathbf{v}$ and the intensive property is the momentum per unit mass, $\beta = d(m\mathbf{v})/dt$, so

$$\sum \mathbf{F} = \frac{d(m\mathbf{v})}{dt} \tag{3.4.2}$$

A lowercase $\mathbf{v}$ is used to denote that this velocity is referenced to the inertial reference frame and to distinguish it from $\mathbf{V}$.

Using the general control volume equation (3.1.13),

$$\frac{dB_{\text{sys}}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho \, d\forall + \int_{CS} \beta\rho \, \mathbf{V} \cdot d\mathbf{A} \tag{3.1.13}$$

and from equation (3.4.2) then

$$\sum \mathbf{F} = \frac{d}{dt}\int_{CV} \mathbf{v}\rho \, d\forall + \int_{CS} \mathbf{v}\rho\mathbf{V} \cdot d\mathbf{A} \tag{3.4.3}$$

which is the *integral momentum equation for fluid flow*. For steady flow, equation (3.4.3) reduces to

$$\sum \mathbf{F} = \int_{CS} \mathbf{v}\rho\,\mathbf{V} \cdot \mathrm{d}\mathbf{A} \tag{3.4.4}$$

When a uniform velocity occurs in the stream crossing the control surface, the integral momentum equation is

$$\sum \mathbf{F} = \frac{d}{dt}\int_{CV} \mathbf{v}\rho \, d\forall + \sum_{CS} \mathbf{v}\rho\mathbf{V} \cdot \mathbf{A} \tag{3.4.5}$$

The momentum can be written for the coordinate directions x, y, and z in the Cartesian coordinate system as

$$\sum F_x = \frac{d}{dt}\int_{CV} v_x \rho \, d\forall + \sum_{CS} v_x (\rho \mathbf{V} \cdot \mathbf{A}) \tag{3.4.6}$$

$$\sum F_y = \frac{d}{dt}\int_{CV} v_y \rho \, d\forall + \sum_{CS} v_y (\rho \mathbf{V} \cdot \mathbf{A}) \tag{3.4.7}$$

$$\sum F_z = \frac{d}{dt}\int_{CV} v_z \rho \, d\forall + \sum_{CS} v_z (\rho \mathbf{V} \cdot \mathbf{A}) \tag{3.4.8}$$

For a steady flow the time derivative in equation (3.4.5) drops out, yielding

$$\sum \mathbf{F} = \sum_{CS} \mathbf{v}\rho \, \mathbf{V} \cdot \mathbf{A} \tag{3.4.9}$$

For a steady flow in which the cross-sectional area of flow does not change along the length of the flow, $\sum_{CS} \mathbf{v}\rho \, \mathbf{V} \cdot \mathbf{A} = 0$, (referred to as uniform flow), equation (3.4.9) reduces to

$$\sum \mathbf{F} = 0 \tag{3.4.10}$$

3.5 PRESSURE AND PRESSURE FORCES IN STATIC FLUIDS

In section 2.4, pressure, absolute pressure, gauge pressure, piezometric head, and pressure force were defined. This section extends that conversation to hydrostatic forces on submerged surfaces and buoyancy.

3.5.1 Hydrostatic Forces

Hydraulic engineers have many engineering applications in which they have to compute the force being exerted on submerged surfaces. The hydrostatic force on any submerged plane surface is equal to the product of the surface area and the pressure acting at the centroid of the plane surface. Consider the force on the plane surface shown in Figure 3.5.1. This plane surface can be divided into an infinite number of differential horizontal planes with width dy and area dA. The distance to the incremental area from the axis O–O is y. The pressure on dA is $p = \gamma \, y \sin\theta$ so that the force dF is $dF = pdA = \gamma \, y \sin\theta \, dA$. The force on the entire submerged plane is obtained by integrating the differential force on the differential area:

$$F = \int_A \gamma y \sin\theta \, dA \tag{3.5.1a}$$

$$= \gamma \sin\theta \int_A y \, dA \tag{3.5.1b}$$

$$= \gamma \sin\theta \, y_c A \tag{3.5.1c}$$

where γ and $\sin\theta$ are constants. The integral $\int_A y\,dA$ is by definition the first moment of the area and $\int_A y(dA/A) = y_c$ is the distance from the O–O axis to the centroid (center of gravity) of the

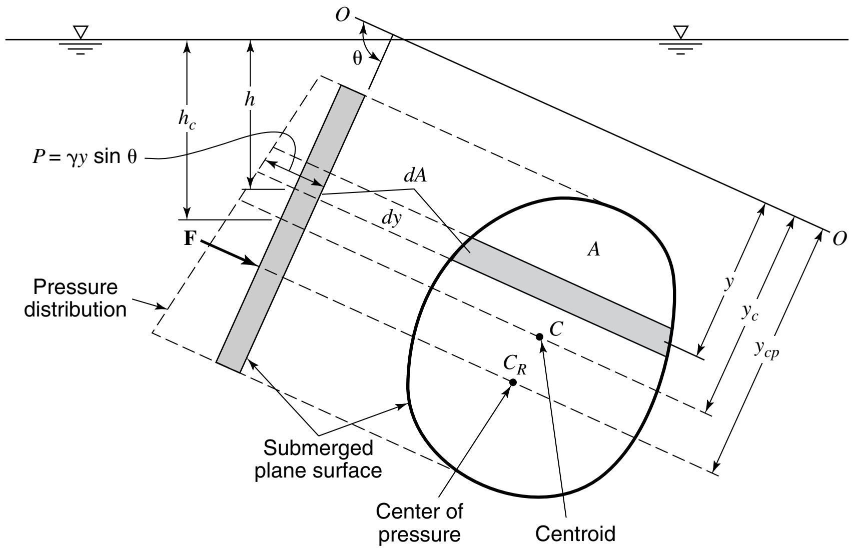

Figure 3.5.1 Hydrostatic pressure on a plane surface.

submerged plane. The vertical distance to the centroid can be defined as $h_c = y_c \sin\theta$, so that the force on the submerged plane is

$$F = \gamma h_c A \tag{3.5.2}$$

Engineers are normally interested in the forces that are in excess of the ambient atmospheric pressures. Keep in mind that atmospheric pressure, for most applications, acts on both sides of the submerged surface so that gauge pressure is of importance.

Even though pressure forces acting on a submerged surface are distributed throughout the surface, engineers are interested in the location of the *center of pressure*, which is the point on the submerged surface where the resultant force acts. The moment equation is

$$y_{cp}F = \int y\,dF \tag{3.5.3}$$

where $dF = p dA$, so

$$y_{cp}F = \int_A yp\,dA \tag{3.5.4}$$

and $p = \gamma y \sin\theta$, so

$$y_{cp}F = \int_A \gamma y^2 \sin\theta\,dA = \gamma \sin\theta \int_A y^2 dA \tag{3.5.5}$$

The integral $\int_A y^2 dA = I_o$ is the *moment of inertia* (*moment of the area*), with respect to an axis formed by the interaction of the plane containing the surface and the free surface. This can also be expressed with respect to the horizontal centroidal axis of the area by the *parallel axis theorem* as

$$I_o = \bar{I} + y_c^2 A \tag{3.5.6}$$

Equation (3.5.5) can now be expressed as

$$y_{cp}F = \gamma \sin\theta\, I_o = \gamma \sin\theta\left(\bar{I} + y_c^2 A\right) \tag{3.5.7}$$

Substituting equation (3.5.1c) and solving for y_{cp} yields

$$y_{cp} = y_c + \frac{\bar{I}}{y_c A} \tag{3.5.8}$$

The vertical distance to the center of pressure h_{cp} is then

$$h_{cp} = y_{cp} \sin \theta \tag{3.5.9}$$

EXAMPLE 3.5.1

Derive the expression for the depth to the center of pressure y_{cp} for a rectangular area ($b \times h$) vertically submerged with the long side (h) at the liquid surface.

SOLUTION

Using equation (3.5.8),

$$y_{cp} = y_c + \frac{\bar{I}}{y_c A}$$

where $y_c = h/2$, $\bar{I} = bh^3/12$, and $A = bh$, we get

$$y_{cp} = \frac{h}{2} + \frac{bh^3/12}{\left(\frac{h}{2}\right)(bh)} = \frac{h}{2} + \frac{h}{6} = \frac{4}{6}h = \frac{2}{3}h$$

EXAMPLE 3.5.2

Determine the hydrostatic force and the location of the center of pressure on the 25 m long dam shown in Figure 3.5.2. The face of the dam is at an angle of 60°. Assume 20°C.

SOLUTION

The diagram in Figure 3.5.2 shows the pressure distribution. Using equation (3.5.2), $h_c = 2.5$ m and $A = (25 \text{ m} \times 5)/\sin 60°$, so the hydrostatic force is

$$F = \gamma h_c A = \left(9.79 \frac{\text{kN}}{\text{m}^3}\right)(2.5 \text{ m})\left(25 \text{ m} \times \frac{5}{\sin 60°} \text{ m}\right)$$

$$= 3.532 \text{ kN}$$

The center of pressure is at 2/3 of the total water depth, $(2/3) \times 5 = 3.33$ m.

EXAMPLE 3.5.3

Consider a vertical rectangular gate ($b = 4$ m and $h = 2$ m) that is vertically submerged in water so that the top of the gate is 4 m below the surface of the water (as shown in Figure 3.5.3). Determine the total resultant force on the gate and the location of the center of pressure.

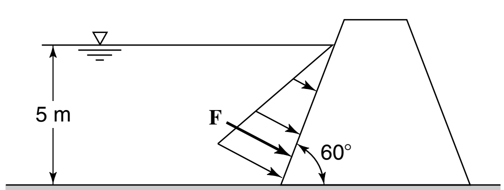

Figure 3.5.2 Hydrostatic force on dam for example 3.5.2.

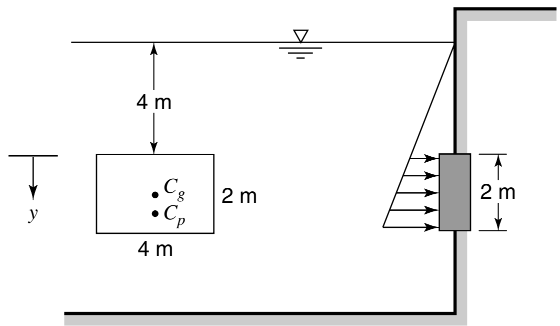

Figure 3.5.3 Vertical rectangular gate for example 3.5.3.

SOLUTION

Use the free-body diagram in Figure 3.5.3. The total resultant force is computed using equation (3.5.2), $F = \gamma h_c A$, where $h_c = 4 + (2/2) = 5$ m:

$$F = (9.79\ \text{kN/m}^3)(5\ \text{m})(4 \times 2\ \text{m}^2) = 396.1\ \text{kN}$$

The location of the center of pressure is computed using equation (3.5.8):

$$y_{cp} = y_c + \frac{\bar{I}}{y_c A} = 5 + \frac{\frac{4 \times 2^3}{12}}{5 \times (4 \times 2)}$$

$$= 5.067\ \text{m}$$

Alternatively, this problem can be solved by the simple integration $F = \int_0^2 \gamma\ h\,dA$, where $dA = 4\ dy$:

$$F = \int_0^2 \gamma h\,\mathrm{d}A = \int_0^2 (9.79)(4 + y)(4\mathrm{d}y)$$

$$= 39.16\left[4y + \frac{y^2}{2}\right]_0^2 = 39.16\left[4 \times 2 + \frac{2^2}{2}\right]$$

$$= 391.6\ \text{kN}$$

EXAMPLE 3.5.4

Consider an inclined rectangular gate with water on one side as shown in Figure 3.5.4. Determine the total resultant force acting on the gate and the location of the center of pressure.

SOLUTION

To determine the total resultant force, $F = \gamma h_c A$, where $h_c = 5 + 1/2(4\cos 60°)$, so that

$$F = (62.4)\left[5 + \frac{1}{2}(4\cos 60°)\right](4 \times 6) = 8{,}986\ \text{lb}$$

The location of the center of pressure is

$$y_{cp} = y_c + \frac{\bar{I}}{y_c A}$$

where $y_c = \dfrac{5}{\cos 60°} + \dfrac{4}{2} = 12$ ft, $\bar{I} = \text{bh}^3/12 = 6 \times 4^3/12 = 32\ \text{ft}^4$ and $A = 6 \times 4 = 24\ \text{ft}^2$:

$$y_{cp} = 12 + \frac{32}{12 \times 24} = 12.11\ \text{ft}$$

Using equation (3.5.9), $h_{cp} = y_{cp} \sin\theta = 12.11\ (\sin 30°) = 6.06$ ft

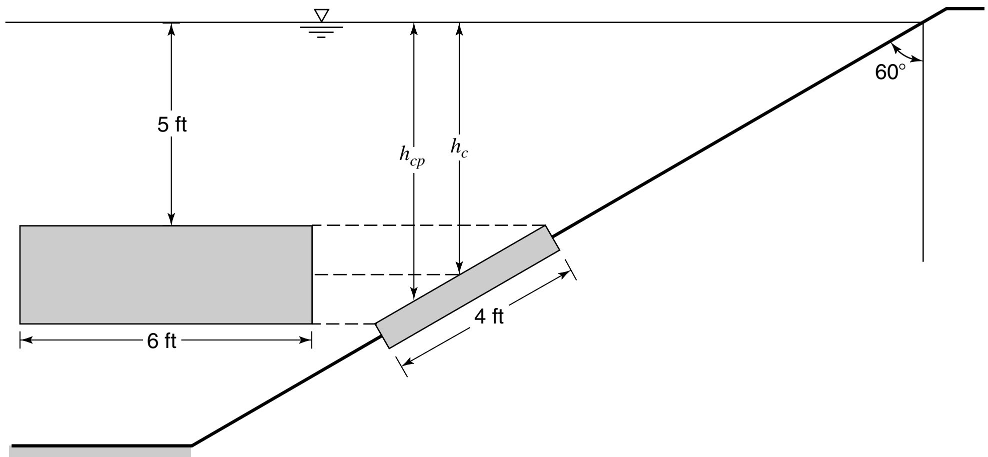

Figure 3.5.4 Inclined rectangular gate for example 3.5.4.

3.5.2 Buoyancy

The submerged body in Figure 3.5.5 is acted upon by gravity and the pressure of the surrounding fluid. On the upper surface of the submerged body, the vertical force is F_y and is equal to the weight of the volume ABCD above the surface. The vertical component of force F'_y on the bottom is the weight of the volume of fluid ABCED. The difference between the two volumes ABCD and ABCED is the volume of the submerged body. Applying the momentum principle, from equation (3.4.7) we get

$$\sum F_y = 0 \tag{3.5.10}$$

The buoyant force F_b is the weight of the volume of fluid DCE and is equal to the weight of the volume of fluid displaced, so that

$$F_b - F'_y + F_y = 0 \tag{3.5.11a}$$

or

$$F_b = F'_y - F_y \tag{3.5.1lb}$$

Archimedes' principle (about 250 B.C.) states that the weight of a submerged body is reduced by an amount equal to the weight of liquid displaced by the body. This principle may be viewed as the difference of vertical pressure forces on the two surfaces DC and DEC. Floating bodies are partially submerged due to the balance of the body weight and buoyancy force.

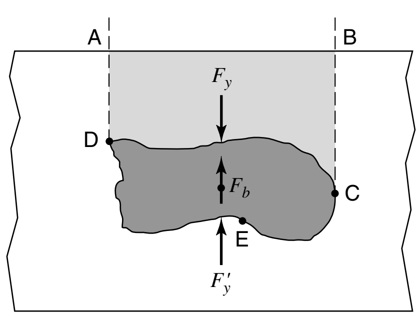

Figure 3.5.5 Forces on a submerged body. Buoyant force, F_b, passes through the centroid of the displaced volume and acts through a point called the center of buoyancy.

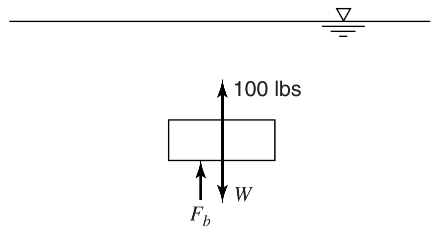

Figure 3.5.6 Free-body diagram for example 3.5.6.

EXAMPLE 3.5.5

A metal block weighs 400 N in air, but when completely submerged in water it weighs 250 N. What is the volume of the metal block?

SOLUTION

Essentially the buoyant force F_b is equal to the weight of water displaced by the metal block, i.e.

$F_b = 400\text{ N} - 250\text{ N} = 150\text{ N}$

The weight $W = (9.79)\,(1000)\,\forall$, where $\forall$ is the volume.

$150 = (9.79)\,(1000)\,\forall$

$\forall = 0.0153\text{ m}^3$

EXAMPLE 3.5.6

An object is 1 ft thick by 1 ft wide by 2 ft long. It weighs 100 lbs at a depth of 10 ft. What is the weight of the object in air and what is its specific gravity?

SOLUTION

Use the free-body diagram in Figure 3.5.6. The summation of forces acting on the object in the vertical direction is

$$\sum F_y = 100 + F_b - W = 0$$

where F_b is the buoyant force and W is the weight of the object.

$$F_b = (62.4\frac{\text{1b}}{\text{ft}^3})(1\text{ ft})(1\text{ ft})(2\text{ ft})$$

$$= 124.8\text{ lbs}$$

$100 + 124.8 - W = 0$

$W = 224.8\text{ lb}$

The specific gravity is $224.8/124.8 = 1.8$.

3.6 VELOCITY DISTRIBUTION

We discussed in section 2.8 that the actual velocity varies throughout a flow section (see Figure 2.8.1 for pipe flow as an illustration). Figure 3.6.1 illustrates velocity profiles in various open-channel flow sections. As a result of these nonuniform velocity distributions in pipe flow and open-channel flow, the velocity head is generally greater than the value computed according to $V^2/2g$ where V is the mean velocity. When using the energy principle, the true velocity head is expressed as $\alpha V^2/2g$, where α is a *kinetic energy correction factor*. Chow (1959) also referred to α as an energy coefficient or *Coriolis coefficient*.

Consider the velocity distribution shown in Figure 2.8.1. The mass of fluid flowing through an area dA per unit time is $(\gamma/g)v\,dA$, where v is the velocity through area dA. The flow of kinetic energy per unit time through this area is $(\gamma/g)\,v\,dA\,(v^2/2) = (\gamma/2g)v^3 dA$. For a known velocity distribution, the total kinetic energy flowing through the section per unit time is

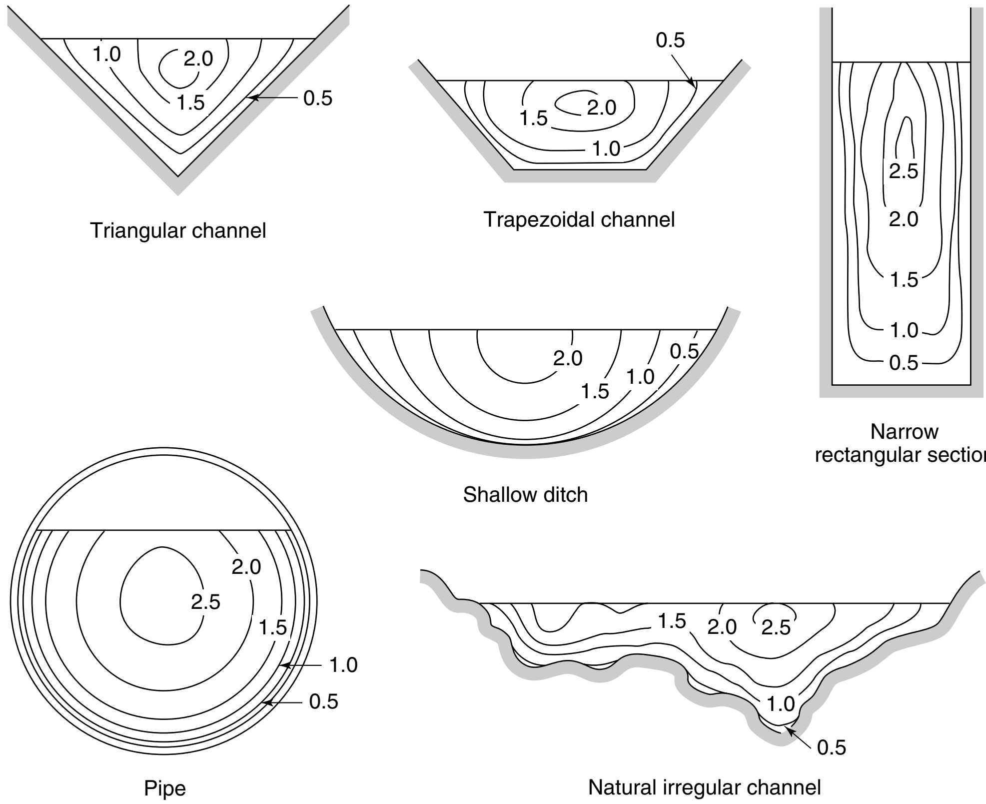

Figure 3.6.1 Typical curves of equal velocity in various channel sections (from Chow (1959)).

$$\underset{\text{(per unit time)}}{\text{Total kinetic energy}} = \frac{\gamma}{2g}\int_A v^3 dA \tag{3.6.1}$$

Using the mean flow velocity V and the coefficient α, the total energy per unit weight is $\alpha V^2/2g$; because the flow across the entire section is γAV. The total kinetic energy transmitted is

$$\begin{aligned}\underset{\text{(per unit time)}}{\text{Total kinetic energy}} &= (\gamma AV)\left(\alpha\frac{V^2}{2g}\right)\\ &= \gamma\alpha A\frac{V^3}{2g}\end{aligned} \tag{3.6.2}$$

From equations (3.6.1) and (3.6.2) we get

$$\frac{\gamma}{2g}\int_A v^3 dA = \gamma\alpha A\frac{V^3}{2g} \tag{3.6.3}$$

and we then solve for the kinetic energy correction factor:

$$\alpha = \frac{1}{AV^3}\int_A v^3 dA \tag{3.6.4}$$

The value of α for flow in circular pipes flowing full with a parabolic velocity distribution is equal to 2 for laminar flow and normally ranges from 1.03 to 1.06 for turbulent flow. Because α is not known precisely, it is not commonly used in pipe flow calculations, and the kinetic energy of fluid per unit weight is $V^2/2g$. The values of α for open-channel flow varies by the type of channel flow. For example, in regular channels, flumes, and spillways, α ranges between 1.10 and 1.20, and for river valleys and areas α ranges between 1.5 and 2.0 with an average of 1.75 (see Chow, 1959).

The nonuniform distribution of velocity also affects the computation of momentum in open-channel flow. The corrected momentum of water passing through a channel section per unit time is

$$\underset{\text{(per unit time)}}{\text{Total momentum}} = \beta_m \frac{\gamma}{g} QV$$

$$= \beta_m \frac{\gamma}{g} AV^2 \tag{3.6.5}$$

where β_m is the *momentum correction factor*, also called the *momentum coefficient* or *Boussinesq coefficient* by Chow (1959). The momentum of water passing through an elemental area dA per unit time is the product of the mass per unit time $(\gamma/g)\, v dA$ and the velocity v, which is $(\gamma/g)\, v^2 dA$. The total momentum of fluid per unit time is

$$\underset{\text{(per unit time)}}{\text{Total momentum}} = \frac{\gamma}{g}\int_A v^2 dA \tag{3.6.6}$$

From equations (3.6.5) and (3.6.6), we get

$$\beta_m \frac{\gamma}{g} AV^2 = \frac{\gamma}{g}\int_A v^2 dA \tag{3.6.7}$$

Solving for the momentum correction factor β_m yields

$$\beta_m = \frac{1}{AV^2}\int_A v^2 dA \tag{3.6.8}$$

According to Chow (1959), the value of β_m for fairly straight prismatic channels varies between 1.01 to 1.12 and for river valleys β_m varies between 1.17 and 1.33.

EXAMPLE 3.6.1

Show that the kinetic energy correction factor is $\alpha = 2$ for laminar flow in a circular pipe.

SOLUTION

In example 2.8.3, the parabolic velocity distribution is expressed as

$$v = v_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right]$$

The rate of $V/v_{\max} = 0.5$ was derived and $A = \pi r^2$. Using equation (3.6.4) yields

$$\alpha = \frac{1}{AV^3}\int_A v^3 dA = \frac{1}{AV^3}\int_0^{r_0} v^3 (2\pi r) dr$$

$$= \frac{1}{\pi r_0^2 (0.5 v_{\max})^3}\int_0^{r_0}\left\{v_{\max}\left[1 - \left(\frac{r}{r_0}\right)^2\right]\right\}^3 (2\pi r) dr$$

$$= \frac{16}{r_0^2}\int_0^{r_0}\left[1 - \left(\frac{r}{r_0}\right)^2\right]^3 r dr = 2$$

PROBLEMS

3.2.1 It is required to reduce a pipe of diameter 8 in to a minimum-diameter pipe that allows the downstream velocity not to exceed twice the upstream velocity. Determine the diameter of the pipe. Assume smooth transition.

3.3.1 Water flows through a pipe of diameter 3 inches. If it is desired to use another pipe for the same flow rate such that the velocity head in the second pipe is four times the velocity head in the first pipe, determine the diameter of the pipe.

3.4.1 If the pipeline in Problem 3.2.1 is horizontal, what is the proportion of the potential energy head at the upstream cross-section that is changed to kinetic energy head at the downstream cross-section? Determine the answer in terms of the discharge.

3.5.1 Derive an expression for the depth to the center of pressure for a triangle of height h and base b that is vertically submerged in water with the vertex at the water surface.

3.5.2 Derive an expression for the depth to the center of pressure for a triangle of height h and base b that is vertically submerged in water with the vertex a distance x below the water surface.

3.5.3 Determine the magnitude and the location of the hydrostatic force on the 2-m by 4-m vertical rectangular gate shown in Figure 3.5.3 if the top of the gate is 6 m below the water surface.

3.5.4 Suppose a vertical flat plate supports water on one side and oil of specific gravity 0.86 on the other side, as shown in Figure P3.5.4. How deep should the oil be so that there is no net horizontal force on the plate? Calculate the moments of the pressure forces about the base of the plate. Are the magnitudes of the moments equal? Why?

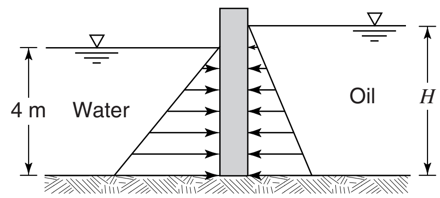

Figure P3.5.4 Vertical flat plate for problem 3.5.4.

3.5.5 Suppose a steel material of specific gravity of 7.8 is attached to a wood of specific gravity 0.8 as shown in Figure P3.5.5 If it is required that the material does not sink or rise when left in static water, what should be the proportion of the volume of the steel to that of the wood in Figure P3.5.5?

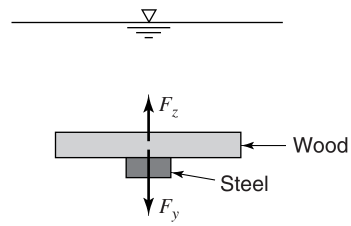

Figure P3.5.5 Problem 3.5.5 System

3.5.6 Rework example 3.5.4 if the top of the inclined rectangular gate is 3 ft below the water surface.

3.6.1 Figure P3.6.1 shows a compound open-channel cross-section. Determine the energy correction factor α. Assume uniform velocities within the subsections.

3.6.2 Determine the momentum correction factor β for Problem 3.6.1.

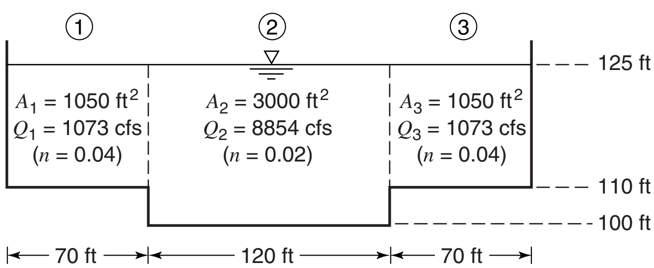

Figure P3.6.1 Compound open-channel cross-section for Problem 3.6.1

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