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## INTRODUCTION

As we continue to step forward into the new millennium with wireless technologies leading the way in which we communicate, it becomes increasingly clear that the dominant consideration in the design of systems employing such technologies will be their ability to perform with adequate margin over a channel perturbed by a host of impairments, not the least of which is multipath fading. This is not to imply that multipath fading channels are something new to be reckoned with; indeed, they have plagued many a system designer for well over 40 years, but rather to serve as a motivation for their ever-increasing significance in the years to come. At the same time, we do not in any way wish to diminish the importance of the fading channel scenarios that occurred well prior to the wireless revolution since indeed many of them still exist and will continue to exist in the future. In fact, it is safe to say that whatever means are developed for dealing with the more sophisticated wireless applications will no doubt also be useful for dealing with the less complicated fading environments of the past.

With the above in mind, what better opportunity is there than now to write a comprehensive book that will provide simple and intuitive solutions to problems dealing with communication system performance evaluation over fading channels? Indeed, as mentioned in the preface, the primary goal of this book is to present a unified method for arriving at a set of tools that will allow the system designer to compute the performance of a host of different digital communication systems characterized by a variety of modulation/detection types and fading channel models. By “set of tools” we mean a compendium of analytical results that not only allow easy yet accurate performance evaluation but at the same time provide insight into the manner in which this performance depends on the key system parameters. To emphasize what was stated above, the set of tools that will be developed in this book are useful not only for the wireless applications that are rapidly filling our current technical journals but also to a host of others involving satellite, terrestrial, and maritime communications.

Our repetitive use of the word “performance” thus far brings us to the purpose of this introductory chapter, namely, to provide several measures of performance related to practical communication system design and to begin exploring the analytical methods by which they may be evaluated. While the deeper meaning of these measures will be truly understood only after their more formal definitions are presented in the chapters that follow, the introduction of these terms here serves to illustrate the various possibilities that exist depending on both need and relative ease of evaluation.

## 1.1 SYSTEM PERFORMANCE MEASURES

### 1.1.1 Average Signal-to-Noise Ratio (SNR)

Probably the most common and well understood performance measure characteristic of a digital communication system is signal-to-noise ratio (SNR). Most often this is measured at the output of the receiver and is thus directly related to the data detection process itself. Of the several possible performance measures that exist, it is typically the easiest to evaluate and most often serves as an excellent indicator of the overall fidelity of the system. While traditionally the term “noise” in signal-to-noise ratio refers to the ever-present thermal noise at the input to the receiver, in the context of a communication system subject to fading impairment, the more appropriate performance measure is *average* SNR, where the term “average” refers to statistical averaging over the probability distribution of the fading. In simple mathematical terms, if  $\gamma$  denotes the instantaneous SNR [a random variable (RV)] at the receiver output that includes the effect of fading, then

$$\bar{\gamma} \triangleq \int_0^{\infty} \gamma p_{\gamma}(\gamma) d\gamma \quad (1.1)$$

is the average SNR, where  $p_{\gamma}(\gamma)$  denotes the probability density function (PDF) of  $\gamma$ . In order to begin to get a feel for what we will shortly describe as a unified approach to performance evaluation, we first rewrite (1.1) in terms of the moment generating function (MGF) associated with  $\gamma$ :

$$M_{\gamma}(s) = \int_0^{\infty} p_{\gamma}(\gamma) e^{s\gamma} d\gamma \quad (1.2)$$

Taking the first derivative of (1.2) with respect to  $s$  and evaluating the result at  $s = 0$ , we immediately see from (1.1) that

$$\bar{\gamma} = \left. \frac{dM_{\gamma}(s)}{ds} \right|_{s=0} \quad (1.3)$$

that is, the ability to evaluate the MGF of the instantaneous SNR (perhaps in closed form) allows immediate evaluation of the average SNR via a simple mathematical operation, namely, differentiation.

To gain further insight into the power of the statement above, we note that in many systems, particularly those dealing with a form of diversity (multichannel) reception known as *maximal-ratio combining* (MRC) (to be discussed in great detail in Chapter 9), the output SNR,  $\gamma$ , is expressed as a sum (combination) of the individual branch (channel) SNRs, namely,  $\gamma = \sum_{l=1}^L \gamma_l$ , where  $L$  denotes the number of channels combined. In addition, it is often reasonable in practice to assume that the channels are independent of each other, that is, that the RVs  $\gamma_l |_{l=1}^L$  are themselves independent. In such instances, the MGF  $M_\gamma(s)$  can be expressed as the product of the MGFs associated with each channel [i.e.,  $M_\gamma(s) = \prod_{l=1}^L M_{\gamma_l}(s)$ ], which as we shall later on in the text can, for a large variety of fading channel statistical models, be computed in closed form.<sup>1</sup> By contrast, even with the assumption of channel independence, the computation of the PDF  $p_\gamma(\gamma)$ , which requires convolution of the various PDFs  $p_{\gamma_l}(\gamma_l) |_{l=1}^L$  that characterize the  $L$  channels, can still be a monumental task. Even in the case where these individual channel PDFs are of the same functional form but are characterized by different average SNRs,  $\bar{\gamma}_l$ , the evaluation of  $p_\gamma(\gamma)$  can still be quite tedious. Such is the power of the MGF-based approach; namely, it circumvents the need for finding the first-order PDF of the output SNR, provided that one is interested in a performance measure that can be expressed in terms of the MGF. Of course, for the case of average SNR, the solution is extremely simple, namely,  $\bar{\gamma} = \sum_{l=1}^L \bar{\gamma}_l$  regardless of whether the channels are independent, and in fact, one never needs to find the MGF at all. However, for other performance measures and also the average SNR of other combining statistics, such as the sum of an ordered set of random variables typical of generalized selection combining (GSC) (to be discussed in Chapter 9), matters are not quite this simple and the points made above for justifying an MGF-based approach are, as we shall see, especially significant.

### 1.1.2 Outage Probability

Another standard performance criterion characteristic of diversity systems operating over fading channels is the so-called *outage probability*—denoted by  $P_{\text{out}}$  and defined as the probability that the instantaneous error probability exceeds a specified value or equivalently the probability that the output SNR,  $\gamma$ , falls below a certain specified threshold,  $\gamma_{\text{th}}$ . Mathematically speaking, we have

$$P_{\text{out}} = \int_0^{\gamma_{\text{th}}} p_\gamma(\gamma) d\gamma \quad (1.4)$$

which is the cumulative distribution function (CDF) of  $\gamma$ , namely,  $P_\gamma(\gamma)$ , evaluated at  $\gamma = \gamma_{\text{th}}$ . Since the PDF and the CDF are related by  $p_\gamma(\gamma) = dP_\gamma(\gamma)/d\gamma$

<sup>1</sup>Note that the existence of the product form for the MGF  $M_\gamma(s)$  does not necessarily imply that the channels are identically distributed; thus, each MGF  $M_{\gamma_l}(s)$  is allowed to maintain its own identity independent of the others. Furthermore, even if the channels are not assumed to be independent, the relation in (1.3) is nevertheless valid and in many instances the MGF of the (combined) output can still be obtained in closed form.

and since  $P_\gamma(0) = 0$ , then the Laplace transforms of these two functions are related by<sup>2</sup>

$$\hat{P}_\gamma(s) = \frac{\hat{p}_\gamma(s)}{s} \quad (1.5)$$

Furthermore, since the MGF is just the Laplace transform of the PDF with argument reversed in sign [i.e.,  $\hat{p}_\gamma(s) = M_\gamma(-s)$ ], then the outage probability can be found from the inverse Laplace transform of the ratio  $M_\gamma(-s)/s$  evaluated at  $\gamma = \gamma_{\text{th}}$

$$P_{\text{out}} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{M_\gamma(-s)}{s} e^{s\gamma_{\text{th}}} ds \quad (1.6)$$

where  $\sigma$  is chosen in the region of convergence of the integral in the complex  $s$  plane. Methods for evaluating inverse Laplace transforms have received widespread attention in the literature. (A good summary of these can be found in the paper by Abate and Whitt [1]). One such numerical technique that is particularly useful for CDFs of positive RVs (such as instantaneous SNR) is discussed in Appendix 9B and applied therein in Chapter 9. For our purpose here, it is sufficient to recognize once again that the evaluation of outage probability can be performed based entirely on the knowledge of the MGF of the output SNR without ever having to compute its PDF.

### 1.1.3 Average Bit Error Probability (BEP)

The third performance criterion and undoubtedly the most difficult of the three to compute is average bit error probability (BEP).<sup>3</sup> On the other hand, it is the one that is most revealing about the nature of the system behavior and the one most often illustrated in documents containing system performance evaluations; thus, it is of primary interest to have a method for its evaluation that reduces the degree of difficulty as much as possible.

The primary reason for the difficulty in evaluating average BEP lies in the fact that the conditional (on the fading) BEP is, in general, a nonlinear function of the instantaneous SNR, as the nature of the nonlinearity is a function of the modulation/detection scheme employed by the system. Thus, for example, in the multichannel case, the average of the conditional BEP over the fading statistics

<sup>2</sup>The symbol “ $\hat{\cdot}$ ” above a function denotes its Laplace transform.

<sup>3</sup>The discussion that follows applies, in principle, equally well to average symbol error probability (SEP). The specific differences between the two are explored in detail in the chapters dealing with system performance. Furthermore, the terms *bit error rate* (BER) and *symbol error rate* (SER) are often used in the literature as alternatives to BEP and SEP. Rather than choose a preference, in this text we shall use these terms interchangeably.

is not a simple average of the per channel performance measure as was true for average SNR. Nevertheless, we shall see momentarily that an MGF-based approach is still quite useful in simplifying the analysis and in a large variety of cases allows unification under a common framework.

Suppose first that the conditional BEP is of the form

$$P_b(E|\gamma) = C_1 \exp(-a_1\gamma) \quad (1.7)$$

such as would be the case for differentially coherent detection of phase-shift-keying (PSK) or noncoherent detection of orthogonal frequency-shift-keying (FSK) (see Chapter 8). Then, the average BEP can be written as

$$P_b(E) \triangleq \int_0^\infty P_b(E|\gamma) p_\gamma(\gamma) d\gamma = \int_0^\infty C_1 \exp(-a_1\gamma) p_\gamma(\gamma) d\gamma = C_1 M_\gamma(-a_1) \quad (1.8)$$

where again  $M_\gamma(s)$  is the MGF of the instantaneous fading SNR and depends only on the fading channel model assumed.

Suppose next that the nonlinear functional relationship between  $P_b(E|\gamma)$  and  $\gamma$  is such that it can be expressed as an integral whose integrand has an exponential dependence on  $\gamma$  in the form of (1.7),<sup>4</sup>

$$P_b(E|\gamma) = \int_{\xi_1}^{\xi_2} C_2 h(\xi) \exp(-a_2 g(\xi)\gamma) d\xi \quad (1.9)$$

where for our purpose here  $h(\xi)$  and  $g(\xi)$  are arbitrary functions of the integration variable and typically both  $\xi_1$  and  $\xi_2$  are finite (although this is not an absolute requirement for what follows).<sup>5</sup> While not at all obvious at this point, suffice it to say that a relationship of the form in (1.9) can result from employing alternative forms of such classic nonlinear functions as the Gaussian  $Q$ -function and Marcum  $Q$ -function (see Chapter 4), which are characteristic of the relationship between  $P_b(E|\gamma)$  and  $\gamma$  corresponding to, for example, coherent detection of PSK and noncoherent detection of quadriphase-shift-keying (QPSK), respectively. Still another possibility is that the nonlinear functional relationship between  $P_b(E|\gamma)$  and  $\gamma$  is inherently in the form of (1.9); thus, no alternative representation need be employed. An example of such occurs for the conditional symbol error probability (SEP) associated with coherent and differentially coherent detection of  $M$ -ary PSK ( $M$ -PSK) (see Chapter 8). Regardless of the particular case at hand,

<sup>4</sup>In the more general case, the conditional BEP might be expressed as a sum of integrals of the type in (1.9).

<sup>5</sup>In principle, (1.9) includes (1.7) as a special case if  $h(\xi)$  is allowed to assume the form of a Dirac delta function located within the interval  $\xi_1 \leq \xi \leq \xi_2$ .

once again averaging (1.9) over the fading gives (after interchanging the order of integration)

$$\begin{aligned}
P_b(E) &= \int_0^\infty P_b(E|\gamma) p_\gamma(\gamma) d\gamma = \int_0^\infty \int_{\xi_1}^{\xi_2} C_2 h(\xi) \exp(-a_2 g(\xi) \gamma) d\xi p_\gamma(\gamma) d\gamma \\
&= C_2 \int_{\xi_1}^{\xi_2} h(\xi) \int_0^\infty \exp(-a_2 g(\xi) \gamma) p_\gamma(\gamma) d\gamma d\xi \\
&= C_2 \int_{\xi_1}^{\xi_2} h(\xi) M_\gamma(-a_2 g(\xi)) d\xi
\end{aligned} \tag{1.10}$$

As we shall see later on in the text, integrals of the form in (1.10) can, for many special cases, be obtained in closed form. At the very worst, with rare exception, the resulting expression will be a single integral with finite limits and an integrand composed of elementary functions.<sup>6</sup> Since (1.8) and (1.10) cover a wide variety of different modulation/detection types and fading channel models, we refer to this approach for evaluating average error probability as *the unified MGF-based approach* and the associated forms of the conditional error probability as the *desired forms*. The first notion of such a unified approach was discussed in Ref. 2 and laid the groundwork for much of the material that follows in this text.

It goes without saying that not every fading channel communication problem fits this description; thus, alternative, but still simple and accurate, techniques are desirable for evaluating system error probability in such circumstances. One class of problems for which a different form of MGF-based approach is possible relates to communication with symmetric binary modulations wherein the decision mechanism constitutes a comparison of a decision variable with a zero threshold. Aside from the obvious uncoded applications, the above-mentioned class also includes the evaluation of pairwise error probability in error-correction-coded systems as discussed in Chapter 12. In mathematical terms, letting  $D|\gamma$  denote the decision variable,<sup>7</sup> then the corresponding conditional BEP is of the form (assuming arbitrarily that a positive data bit was transmitted)

$$P_b(E|\gamma) = \Pr\{D|\gamma < 0\} = \int_{-\infty}^0 p_{D|\gamma}(D) dD = P_{D|\gamma}(0) \tag{1.11}$$

where  $p_{D|\gamma}(D)$  and  $P_{D|\gamma}(D)$  are, respectively, the PDF and CDF of this variable. Aside from the fact that the decision variable  $D|\gamma$  can, in general, take on both positive and negative values whereas the instantaneous fading SNR,  $\gamma$ , is restricted to only positive values, there is a strong resemblance between the binary probability

<sup>6</sup>As we shall see in Chapter 4, the  $h(\xi)$  and  $g(\xi)$  that result from the alternative representations of the Gaussian and Marcum  $Q$ -functions are composed of simple trigonometric functions.

<sup>7</sup>The notation " $D|\gamma$ " is not meant to imply that the decision variable explicitly depends on the fading SNR. Rather, it is merely intended to indicate the dependence of this variable on the fading statistics of the channel. More about this dependence shortly.

of error in (1.11) and the outage probability in (1.4). Thus, by analogy with (1.6), the conditional BEP of (1.11) can be expressed as

$$P_b(E|\gamma) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{M_{D|\gamma}(-s)}{s} ds \quad (1.12)$$

where  $M_{D|\gamma}(-s)$  now denotes the MGF of the decision variable  $D|\gamma$ , that is, the bilateral Laplace transform of  $p_{D|\gamma}(D)$  with argument reversed.

To see how  $M_{D|\gamma}(-s)$  might explicitly depend on  $\gamma$ , we now consider the subclass of problems where the conditional decision variable  $D|\gamma$  corresponds to a quadratic form of independent complex Gaussian RVs, such as a sum of the squared magnitudes of, say,  $L$  independent complex Gaussian RVs—a chi-square RV with  $2L$  degrees of freedom. Such a form occurs for multiple- $(L)$ -channel reception of binary modulations with differentially coherent or noncoherent detection (see Chapter 9). In this instance, the MGF  $M_{D|\gamma}(s)$  happens to be exponential in  $\gamma$  and has the generic form

$$M_{D|\gamma}(s) = f_1(s) \exp(\gamma f_2(s)) \quad (1.13)$$

If as before we let  $\gamma = \sum_{l=1}^L \gamma_l$ , then substituting (1.13) into (1.12) and averaging over the fading results in the average BEP<sup>8</sup>

$$P_b(E) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{M_D(-s)}{s} ds \quad (1.14)$$

where

$$\begin{aligned} M_D(s) &\triangleq \int_0^\infty M_{D|\gamma}(s) p_\gamma(\gamma) d\gamma = f_1(s) \int_0^\infty \exp(\gamma f_2(s)) p_\gamma(\gamma) d\gamma \\ &= f_1(s) M_\gamma(f_2(s)) \end{aligned} \quad (1.15)$$

is the unconditional MGF of the decision variable, which also has the product form

$$M_D(s) = f_1(s) \prod_{l=1}^L M_{\gamma_l}(f_2(s)) \quad (1.16)$$

Finally, by virtue of the fact that the MGF of the decision variable can be expressed in terms of the MGF of the fading variable (SNR) as in (1.15) [or (1.16)], then, analogous to (1.10), we are once again able to evaluate the average BEP solely on the basis of knowledge of the latter MGF.

It is not immediately obvious how to extend the inverse Laplace transform technique discussed in Appendix 9B to CDFs of bilateral RVs; thus other methods

<sup>8</sup>The approach for computing average BEP as described by (1.13) was also described by Biglieri et al. [3] as a unified approach to computing error probabilities over fading channels.

for performing this inversion are required. A number of these, including contour integration using residues, saddle point integration, and numerical integration by Gauss–Chebyshev quadrature rules, are discussed in the literature [3–6] and will be covered later on in the text.

Although the methods dictated by (1.14) and (1.8) or (1.10) cover a wide variety of problems dealing with the performance of digital communication systems over fading channels, there are still some situations that don't lend themselves to either of these two unifying methods. An example of such is the evaluation of the bit error probability performance of an  $M$ -ary noncoherent orthogonal system operating over an  $L$ -path diversity channel (see Chapter 9). However, even in this case there exists an MGF-based approach that greatly simplifies the problem and allows for a result [7] more general than that previously reported by Weng and Leung [8]. We now briefly outline the method, leaving the more detailed treatment to Chapter 9.

Consider an  $M$ -ary communication system where, rather than comparing a single decision variable with a threshold, one decision variable  $U_1 | \gamma$  is compared with the remaining  $M - 1$  decision variables  $U_m$ ,  $m = 2, 3, \dots, M$ , all of which do not depend on the fading statistics.<sup>9</sup> Specifically, a correct symbol decision is made if  $U_1 | \gamma$  is greater than  $U_m$ ,  $m = 2, 3, \dots, M$ . Assuming that the  $M$  decision variables are independent, then, in mathematical terms, the probability of correct decision is given by

$$\begin{aligned} P_s(C | \gamma; u_1) &= \Pr \{ U_2 < u_1, U_3 < u_1, \dots, U_M < u_1 | U_1 | \gamma = u_1 \} \\ &= [\Pr \{ U_2 < u_1 | U_1 | \gamma = u_1 \}]^{M-1} = \left[ \int_0^{u_1} p_{U_2}(u_2) du_2 \right]^{M-1} \\ &= [1 - (1 - P_{U_2}(u_1))]^{M-1} \end{aligned} \quad (1.17)$$

Using the binomial expansion in (1.17), the conditional probability of error  $P_s(E | \gamma; u_1) = 1 - P_s(C | \gamma; u_1)$  can be written as

$$P_s(E | \gamma; u_1) = \sum_{i=1}^{M-1} \binom{M-1}{i} (-1)^{i+1} (1 - P_{U_2}(u_1))^i \triangleq g(u_1) \quad (1.18)$$

Averaging over  $u_1$  and using the Fourier transform relationship between the PDF  $p_{U_1 | \gamma}(u_1)$  and the MGF  $M_{U_1 | \gamma}(j\omega)$ , we obtain

$$\begin{aligned} P_s(E | \gamma) &= \int_0^\infty g(u_1) p_{U_1 | \gamma}(u_1) du_1 \\ &= \int_0^\infty \frac{1}{2\pi} \int_{-\infty}^\infty M_{U_1 | \gamma}(j\omega) e^{-j\omega u_1} g(u_1) d\omega du_1 \end{aligned} \quad (1.19)$$

<sup>9</sup>Again the conditional notation on  $\gamma$  for  $U_1$  is not meant to imply that this decision variable is explicitly a function of the fading SNR but rather to indicate its dependence on the fading statistics.

Again noting that for a noncentral chi-square RV (as is the case for  $U_1 | \gamma$ ) the conditional MGF  $M_{U_1 | \gamma}(j\omega)$  is of the form in (1.13), then averaging (1.19) over  $\gamma$  transforms  $M_{U_1 | \gamma}(j\omega)$  into  $M_{U_1}(j\omega)$  of the form in (1.15), which, when substituted in (1.19) and reversing the order of integration, produces

$$P_s(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(j\omega) M_\gamma(f_2(j\omega)) \left( \int_0^{\infty} e^{-j\omega u_1} g(u_1) du_1 \right) d\omega \quad (1.20)$$

Finally, because the CDF  $P_{U_2}(u_1)$  in (1.18) is that of a central chi-square RV with  $2L$  degrees of freedom, the resulting form of  $g(u_1)$  is such that the integral on  $u_1$  in (1.20) can be obtained in closed form. Thus, as promised, what remains again is an expression for average SEP (which for  $M$ -ary orthogonal signaling can be related to the average BEP by a simple scale factor) whose dependence on the fading statistics is solely through the MGF of the fading SNR.

All the techniques considered thus far for evaluating average error probability performance rely on the ability to evaluate the MGF of the instantaneous fading SNR  $\gamma$ . In dealing with a form of diversity reception referred to as *equal gain combining* (EGC) (to be discussed in great detail in Chapter 9), the instantaneous fading SNR at the output of the combiner takes the form  $\gamma = [(1/\sqrt{L}) \sum_{l=1}^L \sqrt{\gamma_l}]^2$ . In this case, it is more convenient to deal with the MGF of the square root of the instantaneous fading SNR  $x \triangleq \sqrt{\gamma} = (1/\sqrt{L}) \sum_{l=1}^L \sqrt{\gamma_l} = (1/\sqrt{L}) \sum_{l=1}^L x_l$  since if the channels are again assumed independent then again this MGF takes on a product form, namely,  $M_x(s) = \prod_{l=1}^L M_{x_l}(s/\sqrt{L})$ . Since the average BER can alternatively be computed from

$$P_b(E) = \int_0^{\infty} P_b(E|x) p_x(x) dx \quad (1.21)$$

then if, analogous to (1.9),  $P_b(E|x)$  assumes the form

$$P_b(E|x) = \int_{\xi_1}^{\xi_2} C_2 h(\xi) \exp(-a_2 g(\xi) x^2) d\xi \quad (1.22)$$

a variation of the procedure in (1.10) is needed to produce an expression for  $P_b(E)$  in terms of the MGF of  $x$ . First, applying Parseval's theorem [9, p. 27] to (1.21) and letting  $G(j\omega) = \mathcal{F}\{P_b(E|x)\}$  denote the Fourier transform of  $P_b(E|x)$ , then independent of the form of  $P_b(E|x)$ , we obtain<sup>10</sup>

$$\begin{aligned} P_b(E) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) M_x(j\omega) d\omega \\ &= \frac{1}{\pi} \int_0^{\infty} \text{Re}\{G(j\omega) M_x(j\omega)\} d\omega \end{aligned} \quad (1.23)$$

<sup>10</sup>A unified performance evaluation method based on the form of (1.23) and its further simplification in (1.24) has been proposed by Annamalai et al. [14], who refer to their approach as the characteristic function (CHF) method based on Parseval's theorem.

where we have recognized that the imaginary part of the integral must be equal to zero since  $P_b(E)$  is real and that the even part of the integrand is an even function of  $\omega$ . Making the change of variables  $\theta = \tan^{-1} \omega$ , (1.23) can be written in the form of an integral with finite limits:

$$\begin{aligned} P_b(E) &= \frac{1}{\pi} \int_0^{\pi/2} \frac{1}{\cos^2 \theta} \operatorname{Re} \{G(j \tan \theta) M_x(j \tan \theta)\} d\theta \\ &= \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{\sin 2\theta} \operatorname{Re} \{\tan \theta G(j \tan \theta) M_x(j \tan \theta)\} d\theta \end{aligned} \quad (1.24)$$

Now, specifically for the form of  $P_b(E|x)$  in (1.22),  $G(j\omega)$  becomes

$$G(j\omega) = \int_{\xi_1}^{\xi_2} C_2 h(\xi) \int_0^\infty \exp(-a_2 g(\xi) x^2 + j\omega x) dx d\xi \quad (1.25)$$

The inner integral on  $x$  can be evaluated in closed form as

$$\begin{aligned} &\int_0^\infty \exp(-a_2 g(\xi) x^2 + j\omega x) dx \\ &= \frac{1}{2a_2 g(\xi)} \left\{ \sqrt{\pi a_2 g(\xi)} \exp\left(\frac{(j\omega)^2}{4a_2 g(\xi)}\right) + j\omega {}_1F_1\left(1, \frac{3}{2}; \frac{(j\omega)^2}{4a_2 g(\xi)}\right) \right\} \end{aligned} \quad (1.26)$$

where  ${}_1F_1(a, b; c)$  is the confluent hypergeometric function of the first kind [10, p. 1085, Eq. (9.210)]. Therefore, in general, the evaluation of the average BER of (1.24) requires a double integration. However, for a number of specific applications, specifically, particular forms of the functions  $h(\xi)$  and  $g(\xi)$ , the outer integral on  $\xi$  can also be evaluated in closed form; thus, in these instances,  $P_b(E)$  can be obtained as a single integral with finite limits and an integrand involving the MGF of the fading. Methods of error probability evaluation based on the type of MGF approach described above have been considered in the literature [11–13] and will be presented in detail in Chapter 9.

#### 1.1.4 Amount of Fading

The performance measures discussed in Sections 1.1.1–1.1.3 are the ones most commonly employed to describe the behavior of digital communication systems in the presence of fading. Although not as descriptive as the other two, average SNR had the advantage that it was simple to compute in that it required knowledge of only the first statistical moment of the instantaneous SNR. However, in the context of diversity combining, this performance criterion does not capture all the diversity benefits. Indeed, if the diversity advantage were limited to an average SNR gain, then this could be achieved by simply increasing the transmitter power. Of more importance is the aptitude of diversity systems to reduce the fading-induced fluctuations or equivalently in statistical terms, to reduce the relative variance of the

signal envelope that cannot be achieved just by increasing the transmitter power. Thus, in order to capture this effect, we are motivated to look at other performance measures that take into account higher moments of the combiner output SNR. Following along this train of thought, another performance measure that is most often simple to compute and requires knowledge of only the first and second moments of the instantaneous SNR was introduced by the authors [15] when describing the behavior of dual-diversity combining systems over correlated log-normal fading channels. The measure, which is referred to as “amount of fading” (AF), is associated with the output of the combiner and is modeled after a criterion bearing the same name that was originally introduced by Charash [16, p. 29] as a measure of the severity of the fading channel by itself (see Chapter 2, Section 2.2). It is the authors’ suggestion that this same AF measure is often appropriate in the more general context of describing the behavior of systems with arbitrary combining techniques and channel statistics and thus can be used as an alternative performance criterion whenever convenient.<sup>11</sup> Specifically, letting  $\gamma_t$  denote the total instantaneous SNR at the combiner output, we define AF by

$$\text{AF} = \frac{\text{var } \gamma_t}{(E[\gamma_t])^2} = \frac{E(\gamma_t^2) - (E[\gamma_t])^2}{(E[\gamma_t])^2} \quad (1.27)$$

which can be expressed in terms of the MGF of  $\gamma_t$  by

$$\text{AF} = \frac{\left. \frac{d^2 M_{\gamma_t}(s)}{ds^2} \right|_{s=0} - \left( \left. \frac{d M_{\gamma_t}(s)}{ds} \right|_{s=0} \right)^2}{\left( \left. \frac{d M_{\gamma_t}(s)}{ds} \right|_{s=0} \right)^2} \quad (1.28)$$

Because the AF defined in (1.27) is computed at the output of the combiner, its evaluation will reflect the behavior of the particular diversity combining technique as well as the statistics of the fading channel and thus, as mentioned above, is a measure of the performance of the entire system. Closed-form expressions for a variety of such evaluations will be presented in Chapter 9.

### 1.1.5 Average Outage Duration

In certain communication system applications such as adaptive transmission schemes, the performance metrics discussed above do not provide enough information for the overall system design and configuration. In that case, in addition to these performance measures, the frequency of outages and the average outage duration (AOD) (also known as the “average fade duration”) are important performance criteria for the proper selection of the transmission symbol rate, interleaver depth, packet length, and/or time slot duration.

<sup>11</sup>Perhaps the earliest indication of such a performance measure appears in a paper by Win and Winters [17], who described the behavior of hybrid selection combining/maximal-ratio combining in the presence of Rayleigh fading in terms of the normalized standard deviation (NSD) of the diversity combiner output SNR, which coincidentally is equal to the square root of the amount of fading.

As discussed above, in purely noise-limited systems, an outage is declared whenever the output SNR,  $\gamma$ , falls below a predetermined threshold  $\gamma_{\text{th}}$ , (i.e.,  $\gamma < \gamma_{\text{th}}$ ). The AOD,  $T(\gamma_{\text{th}})$  (in seconds) is a measure of how long, on the average, the system remains in the outage state. Mathematically speaking, the AOD is well known to be given by [18]

$$T(\gamma_{\text{th}}) = \frac{P_{\text{out}}}{N(\gamma_{\text{th}})} \quad (1.29)$$

where  $P_{\text{out}}$  was defined and discussed in Section 1.1.2 and  $N(\gamma_{\text{th}})$  is the frequency of outages or equivalently the average level crossing rate (LCR) of the output SNR  $\gamma$  at level  $\gamma_{\text{th}}$ , which can be obtained from the well-known Rice formula [19]

$$N(\gamma_{\text{th}}) = \int_0^{\infty} \dot{\gamma} f_{\gamma, \dot{\gamma}}(\gamma_{\text{th}}, \dot{\gamma}) d\dot{\gamma}, \quad (1.30)$$

with  $f_{\gamma, \dot{\gamma}}(\gamma, \dot{\gamma})$  the joint PDF of  $\gamma$  and its time derivative  $\dot{\gamma}$ . Methods and analytical expressions for the evaluation of the average LCR, and thus AOD, for various diversity combining schemes will be presented in Chapter 9.

## 1.2 CONCLUSIONS

Without regard to the specific application or performance measure, we have briefly demonstrated in this chapter that for a wide variety of digital communication systems covering virtually all known modulation/detection techniques and practical fading channel models, there exists an MGF-based approach that simplifies the evaluation of this performance. In the biggest number of these instances, the MGF-based approach is encompassed in a unified framework that allows the development of a set of generic tools to replace the case-by-case analyses typical of previous contributions in the literature. It is the authors' hope that by the time the readers reach the end of this book and have experienced the exhaustive set of practical circumstances where these tools are useful, they will fully appreciate the power behind the MGF-based approach and as such will generate for themselves an insight into finding new and exciting applications.

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