

Contents

Preface XV

Introduction 1

Bibliography 9

1	Review of Classical Mechanics and String Field Theory	11
1.1	Preview and Rationale	11
1.2	Review of Lagrangians and Hamiltonians	13
1.2.1	Hamilton's Equations in Multiple Dimensions	14
1.3	Derivation of the Lagrange Equation from Hamilton's Principle	16
1.4	Linear, Multiparticle Systems	18
1.4.1	The Laplace Transform Method	23
1.4.2	Damped and Driven Simple Harmonic Motion	24
1.4.3	Conservation of Momentum and Energy	26
1.5	Effective Potential and the Kepler Problem	26
1.6	Multiparticle Systems	29
1.7	Longitudinal Oscillation of a Beaded String	32
1.7.1	Monofrequency Excitation	33
1.7.2	The Continuum Limit	34
1.8	Field Theoretical Treatment and Lagrangian Density	36
1.9	Hamiltonian Density for Transverse String Motion	39
1.10	String Motion Expressed as Propagating and Reflecting Waves	40
1.11	Problems	42
	Bibliography	44
2	Geometry of Mechanics, I, Linear	45
2.1	Pairs of Planes as Covariant Vectors	47
2.2	Differential Forms	53
2.2.1	Geometric Interpretation	53
2.2.2	Calculus of Differential Forms	57
2.2.3	Familiar Physics Equations Expressed Using Differential Forms	61

2.3	Algebraic Tensors	66
2.3.1	Vectors and Their Duals	66
2.3.2	Transformation of Coordinates	68
2.3.3	Transformation of Distributions	72
2.3.4	Multi-index Tensors and their Contraction	73
2.3.5	Representation of a Vector as a Differential Operator	76
2.4	(Possibly Complex) Cartesian Vectors in Metric Geometry	79
2.4.1	Euclidean Vectors	79
2.4.2	Skew Coordinate Frames	81
2.4.3	Reduction of a Quadratic Form to a Sum or Difference of Squares	81
2.4.4	Introduction of Covariant Components	83
2.4.5	The Reciprocal Basis	84
	Bibliography	86
3	Geometry of Mechanics, II, Curvilinear	89
3.1	(Real) Curvilinear Coordinates in n -Dimensions	90
3.1.1	The Metric Tensor	90
3.1.2	Relating Coordinate Systems at Different Points in Space	92
3.1.3	The Covariant (or Absolute) Differential	97
3.2	Derivation of the Lagrange Equations from the Absolute Differential	102
3.2.1	Practical Evaluation of the Christoffel Symbols	108
3.3	Intrinsic Derivatives and the Bilinear Covariant	109
3.4	The Lie Derivative – Coordinate Approach	111
3.4.1	Lie-Dragged Coordinate Systems	111
3.4.2	Lie Derivatives of Scalars and Vectors	115
3.5	The Lie Derivative – Lie Algebraic Approach	120
3.5.1	Exponential Representation of Parameterized Curves	120
3.6	Identification of Vector Fields with Differential Operators	121
3.6.1	Loop Defect	122
3.7	Coordinate Congruences	123
3.8	Lie-Dragged Congruences and the Lie Derivative	125
3.9	Commutators of Quasi-Basis-Vectors	130
	Bibliography	132
4	Geometry of Mechanics, III, Multilinear	133
4.1	Generalized Euclidean Rotations and Reflections	133
4.1.1	Reflections	134
4.1.2	Expressing a Rotation as a Product of Reflections	135
4.1.3	The Lie Group of Rotations	136
4.2	Multivectors	138

4.2.1	Volume Determined by 3- and by n -Vectors	138
4.2.2	Bivectors	140
4.2.3	Multivectors and Generalization to Higher Dimensionality	141
4.2.4	Local Radius of Curvature of a Particle Orbit	143
4.2.5	“Supplementary” Multivectors	144
4.2.6	Sums of p -Vectors	145
4.2.7	Bivectors and Infinitesimal Rotations	145
4.3	Curvilinear Coordinates in Euclidean Geometry (Continued)	148
4.3.1	Repeated Exterior Derivatives	148
4.3.2	The Gradient Formula of Vector Analysis	149
4.3.3	Vector Calculus Expressed by Differential Forms	151
4.3.4	Derivation of Vector Integral Formulas	154
4.3.5	Generalized Divergence and Gauss’s Theorem	157
4.3.6	Metric-Free Definition of the “Divergence” of a Vector	159
4.4	Spinors in Three-Dimensional Space	161
4.4.1	Definition of Spinors	162
4.4.2	Demonstration that a Spinor is a Euclidean Tensor	162
4.4.3	Associating a 2×2 Reflection (Rotation) Matrix with a Vector (Bivector)	163
4.4.4	Associating a Matrix with a Trivector (Triple Product)	164
4.4.5	Representations of Reflections	164
4.4.6	Representations of Rotations	165
4.4.7	Operations on Spinors	166
4.4.8	Real Euclidean Space	167
4.4.9	Real Pseudo-Euclidean Space	167
	Bibliography	167
5	Lagrange–Poincaré Description of Mechanics	169
5.1	The Poincaré Equation	169
5.1.1	Some Features of the Poincaré Equations	179
5.1.2	Invariance of the Poincaré Equation	180
5.1.3	Translation into the Language of Forms and Vector Fields	182
5.1.4	Example: Free Motion of a Rigid Body with One Point Fixed	183
5.2	Variational Derivation of the Poincaré Equation	186
5.3	Restricting the Poincaré Equation With Group Theory	189
5.3.1	Continuous Transformation Groups	189
5.3.2	Use of Infinitesimal Group Parameters as Quasicoordinates	193
5.3.3	Infinitesimal Group Operators	195
5.3.4	Commutation Relations and Structure Constants of the Group	199
5.3.5	Qualitative Aspects of Infinitesimal Generators	201
5.3.6	The Poincaré Equation in Terms of Group Generators	204
5.3.7	The Rigid Body Subject to Force and Torque	206
	Bibliography	217

6	Newtonian/Gauge Invariant Mechanics	219
6.1	Vector Mechanics	219
6.1.1	Vector Description in Curvilinear Coordinates	219
6.1.2	The Frenet–Serret Formulas	222
6.1.3	Vector Description in an Accelerating Coordinate Frame	226
6.1.4	Exploiting the Fictitious Force Description	232
6.2	Single Particle Equations in Gauge Invariant Form	238
6.2.1	Newton’s Force Equation in Gauge Invariant Form	239
6.2.2	Active Interpretation of the Transformations	242
6.2.3	Newton’s Torque Equation	246
6.2.4	The Plumb Bob	248
6.3	Gauge Invariant Description of Rigid Body Motion	252
6.3.1	Space and Body Frames of Reference	253
6.3.2	Review of the Association of 2×2 Matrices to Vectors	256
6.3.3	“Association” of 3×3 Matrices to Vectors	258
6.3.4	Derivation of the Rigid Body Equations	259
6.3.5	The Euler Equations for a Rigid Body	261
6.4	The Foucault Pendulum	262
6.4.1	Fictitious Force Solution	263
6.4.2	Gauge Invariant Solution	265
6.4.3	“Parallel” Translation of Coordinate Axes	270
6.5	Tumblers and Divers	274
	Bibliography	276
7	Hamiltonian Treatment of Geometric Optics	277
7.1	Analogy Between Mechanics and Geometric Optics	278
7.1.1	Scalar Wave Equation	279
7.1.2	The Eikonal Equation	281
7.1.3	Determination of Rays from Wavefronts	282
7.1.4	The Ray Equation in Geometric Optics	283
7.2	Variational Principles	285
7.2.1	The Lagrange Integral Invariant and Snell’s Law	285
7.2.2	The Principle of Least Time	287
7.3	Paraxial Optics, Gaussian Optics, Matrix Optics	288
7.4	Huygens’ Principle	292
	Bibliography	294
8	Hamilton–Jacobi Theory	295
8.1	Hamilton–Jacobi Theory Derived from Hamilton’s Principle	295
8.1.1	The Geometric Picture	297
8.1.2	Constant S Wavefronts	298
8.2	Trajectory Determination Using the Hamilton–Jacobi Equation	299

8.2.1	Complete Integral	299
8.2.2	Finding a Complete Integral by Separation of Variables	300
8.2.3	Hamilton–Jacobi Analysis of Projectile Motion	301
8.2.4	The Jacobi Method for Exploiting a Complete Integral	302
8.2.5	Completion of Projectile Example	304
8.2.6	The Time-Independent Hamilton–Jacobi Equation	305
8.2.7	Hamilton–Jacobi Treatment of 1D Simple Harmonic Motion	306
8.3	The Kepler Problem	307
8.3.1	Coordinate Frames	308
8.3.2	Orbit Elements	309
8.3.3	Hamilton–Jacobi Formulation.	310
8.4	Analogies Between Optics and Quantum Mechanics	314
8.4.1	Classical Limit of the Schrödinger Equation	314
	Bibliography	316
9	Relativistic Mechanics	317
9.1	Relativistic Kinematics	317
9.1.1	Form Invariance	317
9.1.2	World Points and Intervals	318
9.1.3	Proper Time	319
9.1.4	The Lorentz Transformation	321
9.1.5	Transformation of Velocities	322
9.1.6	4-Vectors and Tensors	322
9.1.7	Three-Index Antisymmetric Tensor	325
9.1.8	Antisymmetric 4-Tensors	325
9.1.9	The 4-Gradient, 4-Velocity, and 4-Acceleration	326
9.2	Relativistic Mechanics	327
9.2.1	The Relativistic Principle of Least Action	327
9.2.2	Energy and Momentum	328
9.2.3	4-Vector Notation	329
9.2.4	Forced Motion	329
9.2.5	Hamilton–Jacobi Formulation	330
9.3	Introduction of Electromagnetic Forces into Relativistic Mechanics	332
9.3.1	Generalization of the Action	332
9.3.2	Derivation of the Lorentz Force Law	334
9.3.3	Gauge Invariance	335
	Bibliography	338
10	Conservation Laws and Symmetry	339
10.1	Conservation of Linear Momentum	339
10.2	Rate of Change of Angular Momentum: Poincaré Approach	341

10.3	Conservation of Angular Momentum: Lagrangian Approach	342
10.4	Conservation of Energy	343
10.5	Cyclic Coordinates and Routhian Reduction	344
10.5.1	Integrability; Generalization of Cyclic Variables	347
10.6	Noether's Theorem	348
10.7	Conservation Laws in Field Theory	352
10.7.1	Ignorable Coordinates and the Energy Momentum Tensor	352
10.8	Transition From Discrete to Continuous Representation	356
10.8.1	The 4-Current Density and Charge Conservation	356
10.8.2	Energy and Momentum Densities	360
10.9	Angular Momentum of a System of Particles	362
10.10	Angular Momentum of a Field	363
	Bibliography	364
11	Electromagnetic Theory	365
11.1	The Electromagnetic Field Tensor	367
11.1.1	The Lorentz Force Equation in Tensor Notation	367
11.1.2	Lorentz Transformation and Invariants of the Fields	369
11.2	The Electromagnetic Field Equations	370
11.2.1	The Homogeneous Pair of Maxwell Equations	370
11.2.2	The Action for the Field, Particle System	370
11.2.3	The Electromagnetic Wave Equation	372
11.2.4	The Inhomogeneous Pair of Maxwell Equations	373
11.2.5	Energy Density, Energy Flux, and the Maxwell Stress Energy Tensor	374
	Bibliography	377
12	Relativistic Strings	379
12.1	Introduction	379
12.1.1	Is String Theory Appropriate?	379
12.1.2	Parameterization Invariance	381
12.1.3	Postulating a String Lagrangian	381
12.2	Area Representation in Terms of the Metric	383
12.3	The Lagrangian Density and Action for Strings	384
12.3.1	A Revised Metric	384
12.3.2	Parameterization of String World Surface by σ and τ	385
12.3.3	The Nambu-Goto Action	385
12.3.4	String Tension and Mass Density	387
12.4	Equations of Motion, Boundary Conditions, and Unexcited Strings	389
12.5	The Action in Terms of Transverse Velocity	391
12.6	Orthogonal Parameterization by Energy Content	394

12.7	General Motion of a Free Open String	396
12.8	A Rotating Straight String	398
12.9	Conserved Momenta of a String	400
12.9.1	Angular Momentum of Uniformly Rotating Straight String	401
12.10	Light Cone Coordinates	402
12.11	Oscillation Modes of a Relativistic String	406
	Bibliography	408
13	General Relativity	409
13.1	Introduction	409
13.2	Transformation to Locally Inertial Coordinates	412
13.3	Parallel Transport on a Surface	413
13.3.1	Geodesic Curves	416
13.4	The Twin Paradox in General Relativity	417
13.5	The Curvature Tensor	422
13.5.1	Properties of Curvature Tensor, Ricci Tensor, and Scalar Curvature	423
13.6	The Lagrangian of General Relativity and the Energy–Momentum Tensor	425
13.7	“Derivation” of the Einstein Equation	428
13.8	Weak, Nonrelativistic Gravity	430
13.9	The Schwarzschild Metric	433
13.9.1	Orbit of a Particle Subject to the Schwarzschild Metric	434
13.10	Gravitational Lensing and Red Shifts	437
	Bibliography	440
14	Analytic Bases for Approximation	441
14.1	Canonical Transformations	441
14.1.1	The Action as a Generator of Canonical Transformations	441
14.2	Time-Independent Canonical Transformation	446
14.3	Action-Angle Variables	448
14.3.1	The Action Variable of a Simple Harmonic Oscillator	448
14.3.2	Adiabatic Invariance of the Action I	449
14.3.3	Action/Angle Conjugate Variables	453
14.3.4	Parametrically Driven Simple Harmonic Motion	455
14.4	Examples of Adiabatic Invariance	457
14.4.1	Variable Length Pendulum	457
14.4.2	Charged Particle in Magnetic Field	459
14.4.3	Charged Particle in a Magnetic Trap	461
14.5	Accuracy of Conservation of Adiabatic Invariants	466
14.6	Conditionally Periodic Motion	469
14.6.1	Stäckel’s Theorem	470

14.6.2	Angle Variables	471
14.6.3	Action/Angle Coordinates for Keplerian Satellites	474
	Bibliography	475
15	Linear Hamiltonian Systems	477
15.1	Linear Hamiltonian Systems	477
15.1.1	Inhomogeneous Equations	479
15.1.2	Exponentiation, Diagonalization, and Logarithm Formation of Matrices	479
15.1.3	Alternate Coordinate Ordering	481
15.1.4	Eigensolutions	481
15.2	Periodic Linear Systems	484
15.2.1	Floquet's Theorem	485
15.2.2	Lyapunov's Theorem	487
15.2.3	Characteristic Multipliers, Characteristic Exponents	487
15.2.4	The Variational Equations	489
	Bibliography	490
16	Perturbation Theory	491
16.1	The Lagrange Planetary Equations	492
16.1.1	Derivation of the Equations	492
16.1.2	Relation Between Lagrange and Poisson Brackets	496
16.2	Advance of Perihelion of Mercury	497
16.3	Iterative Analysis of Anharmonic Oscillations	502
16.4	The Method of Krylov and Bogoliubov	508
16.4.1	First Approximation	508
16.4.2	Equivalent Linearization	512
16.4.3	Power Balance, Harmonic Balance	514
16.4.4	Qualitative Analysis of Autonomous Oscillators	515
16.4.5	Higher K-B Approximation	518
16.5	Superconvergent Perturbation Theory	523
16.5.1	Canonical Perturbation Theory	523
16.5.2	Application to Gravity Pendulum	525
16.5.3	Superconvergence	527
	Bibliography	527
17	Symplectic Mechanics	529
17.1	The Symplectic Properties of Phase Space	530
17.1.1	The Canonical Momentum 1-Form	530
17.1.2	The Symplectic 2-Form $\tilde{\omega}$	533
17.1.3	Invariance of the Symplectic 2-Form	537
17.1.4	Use of $\tilde{\omega}$ to Associate Vectors and 1-Forms	538

17.1.5	Explicit Evaluation of Some Inner Products	539
17.1.6	The Vector Field Associated with $\widetilde{\mathbf{dH}}$	540
17.1.7	Hamilton's Equations in Matrix Form	541
17.2	Symplectic Geometry	543
17.2.1	Symplectic Products and Symplectic Bases	543
17.2.2	Symplectic Transformations	545
17.2.3	Properties of Symplectic Matrices	546
17.3	Poisson Brackets of Scalar Functions	554
17.3.1	The Poisson Bracket of Two Scalar Functions	554
17.3.2	Properties of Poisson Brackets	555
17.3.3	The Poisson Bracket and Quantum Mechanics	555
17.4	Integral Invariants	557
17.4.1	Integral Invariants in Electricity and Magnetism	557
17.4.2	The Poincaré–Cartan Integral Invariant	560
17.5	Invariance of the Poincaré–Cartan Integral Invariant I.I.	562
17.5.1	The Extended Phase Space 2-Form and its Special Eigenvector	563
17.5.2	Proof of Invariance of the Poincaré Relative Integral Invariant	565
17.6	Symplectic System Evolution	566
17.6.1	Liouville's Theorem and Generalizations	568
	Bibliography	570
	Index	571

