Limits and Continuity

The single major concept that separates precalculus math from calculus is that of the limit of a function. Although none of the formal definitions of limits are on the AP exam, you must have a solid, intuitive understanding of how a limit works in order to be able to apply limits to concepts such as continuity. You must also master a variety of different techniques used to find limits, because the AP exam will include multiple-choice questions that deal directly with finding limits.

Intuitive Definition of a Limit

Perhaps the simplest way to understand the concept of the limit of a function is through a graphical interpretation and in terms of the following translation from calculus to English:

Calculus	$\Leftrightarrow$	English
$\lim_{x \to 2}(2x-1)=?$	$\Leftrightarrow$	Read: "What is the limit of $2x - 1$ as x approaches 2?" Think: As the x-coordinates get closer and closer to 2 what values (if any) are the y-coordinates getting closer to?

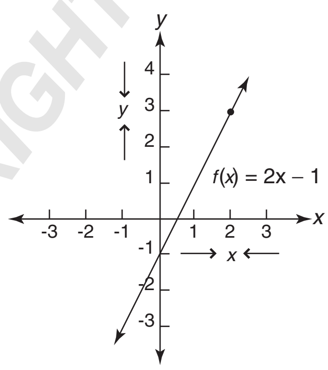

Figure 2.1

From the graph, it should be clear that as the x-coordinates get closer and closer to 2, the y-coordinates get closer and closer to 3. The calculus notation for this is

$$\lim_{x \to 2}(2x-1)=3$$

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The value of the function at the exact point where $x = 2$ is irrelevant to answering the limit question. The limit question is answered by determining what value the y-coordinates are approaching as x gets *closer and closer* to 2. Examine the following table of values.

x	1	1.5	1.6	1.8	1.9	1.99		2.01	2.1	2.2	2.3	2.5	3
$f(x)$	1	2	2.2	2.6	2.8	2.98		3.02	3.2	3.4	3.6	4	5

On the top line, x-coordinates have been selected as a series of values approaching 2 from both directions — that is, increasing from 1 and decreasing from 3. Note that the corresponding y-values on the bottom line are approaching 3 from both directions, that is, increasing from 1 and decreasing from 5. The ordered pair (2, 3) is not included in the chart because it is not relevant to determining the limit. In fact, if a function $g(x)$ is created that has the same values as $f(x)$ above, but is defined differently (or even undefined) at $x = 2$, the limit of $g(x)$ is the same as for $f(x)$:

$$g(x) = \begin{cases} 2x - 1 & \text{for } x \neq 2 \\ 2 & \text{for } x = 2 \end{cases}$$

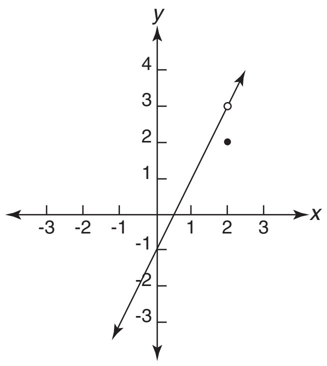

Figure 2.2

From the graph it should be clear that here, too, as the x-coordinates get closer and closer to 2, the y-coordinates get closer and closer to 3. In the language of calculus

$$\lim_{x \to 2} g(x) = 3$$

Here are some other examples of limits:

$$\lim_{x \to 3} |6 - 3x| = 3$$

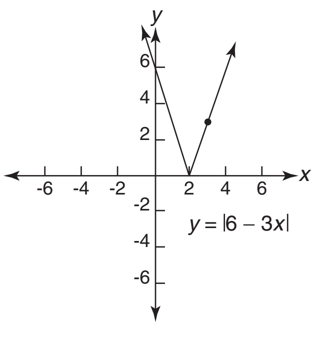

Figure 2.3

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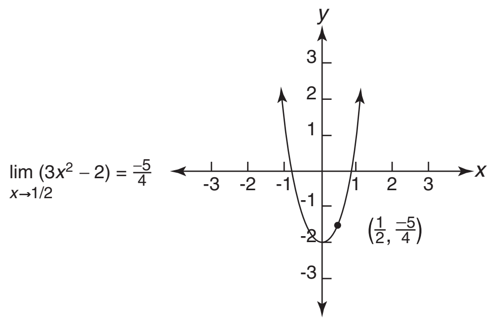

Figure 2.4

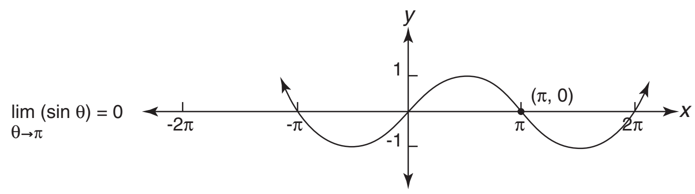

Figure 2.5

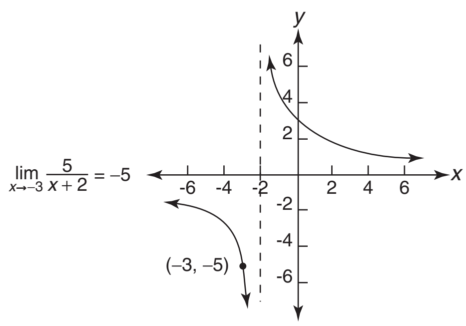

Figure 2.6

Generally, finding the limit of a continuous function requires straightforward substitution. Even the limits of certain types of discontinuous functions, such as $g(x)$ above, can be found by direct substitution. However, some types of discontinuities may dramatically affect the limit of the function at the point of discontinuity, as shown in the examples that follow.

Sample

What is $\lim_{x \to 1} \frac{2}{(x-1)^2}$?

Begin by sketching the graph. The graph shows that as the x-coordinates get closer and closer to 1, the function values (y-coordinates) simply get bigger and bigger. The calculus notation used to symbolize this is

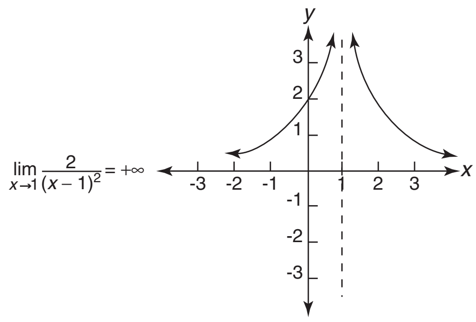

Figure 2.7

Caution: This notation is not meant to imply that a limit actually exists. Remember that ∞ (infinity) is a concept, not a number. The notation used above, whereby the limit "equals infinity," is generally preferred because it indicates the behavior of the graph. But it is also correct to write

$$\lim_{x \to 1} \frac{2}{(x-1)^2} \text{ does not exist}$$

Sample

Find $\lim_{x \to -1} h(x)$ if $h(x) = \begin{cases} -2x-4 & \text{for } x \le -1 \\ x^2 & \text{for } x > -1 \end{cases}$

Again, begin by sketching the graph. Examine the *y*-coordinates as the *x*-coordinates get closer and closer to –1. There is no single number that the function values are getting closer to: From the left the *y*-values get closer and closer to –2, and from the right they get closer and closer to 1. This implies that $\lim_{x \to -1} h(x)$ does not exist.

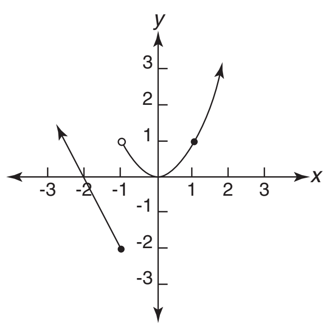

Figure 2.8

The limit notation can be modified to symbolize the situation where the lefthand and righthand limits are different.

$$\lim_{x \to -1^-} h(x) = -2 \text{ and } \lim_{x \to -1^+} h(x) = 1$$

Such **one-sided limits** will be explained in greater detail later in this chapter.

Algebraic Techniques for Finding Limits

Finding the limit of a continuous function requires the simple algebraic technique of substitution. For more complicated functions, other algebraic techniques are required. These techniques include

1. Substitution
2. Simplifying expressions
3. Rationalizing the numerator or denominator

These last two techniques are used when direct substitution yields the **indeterminate form,** wherein both numerator and denominator are zero (0/0). When the indeterminate form arises from direct substitution, some type of algebra must be done to change the form of the function. *Caution: Do not make the mistake of assuming that the indeterminate form of* 0/0 *somehow "cancels out" and is equal to 1.* This is not true, as shown by the following examples.

Sample

What is $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1}$?

Substituting $x = 1$ in directly gives

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1} = \frac{1 + 1 - 2}{1 - 1} = \frac{0}{0}$$

This is the indeterminate form, which implies that algebra needs to be done — in this case, factoring.

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1} = \lim_{x \to 1} \frac{(x+2)(x-1)}{(x+1)(x-1)} \qquad \text{by factoring}$$

$$= \lim_{x \to 1} \left(\frac{x+2}{x+1}\right) \qquad \text{by canceling}$$

$$= 3/2 \qquad \text{by substitution}$$

The graph of $f(x)$, a rational function, shows a hole at (1, 3/2). Recall that the limit is not affected by an undefined or unusual value at the limit site.

$$f(x) = \frac{x^2 + x - 2}{x^2 - 1}$$
$$= \frac{x+2}{x+1} \quad x \neq 1$$

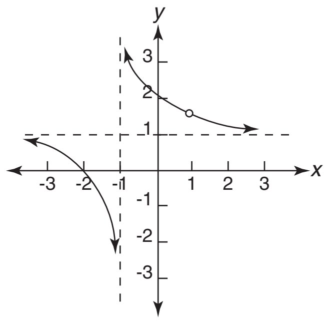

Figure 2.9

Sample

What is $\lim\limits_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$?

Again, direct substitution yields the indeterminate form.

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{4} - 2}{4 - 4} = \frac{0}{0}$$

Therefore, you must do algebra to change the form.

Rationalizing the numerator,	$\lim\limits_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}\left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2}\right)$
Simplifying,	$= \lim\limits_{x \to 4} \frac{x - 4}{(x - 4)\left(\sqrt{x} + 2\right)}$
Canceling factors,	$= \lim\limits_{x \to 4} \left(\frac{1}{\sqrt{x} + 2}\right)$
Substituting,	$= \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$

One-Sided Limits

One-sided limits are used most frequently with piece functions, functions with domain restrictions, or infinite limits. One-sided limits can be interpreted similarly to general limits:

Calculus	⇔	English
$\lim\limits_{x \to 1^+} (x^2) = ?$	⇔	Think: As the x-coordinates get closer and closer to 1 from the right side, what values (if any) are the y-coordinates getting closer to?

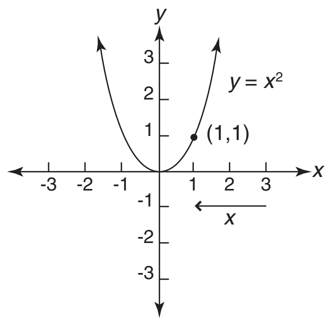

Figure 2.10

From the graph, it should be clear that the *y*-coordinates are approaching 1 as the *x*-coordinates approach 1 from the right. The calculus notation is

$$\lim_{x \to 1^+}\left(x^2\right) = 1$$

Similarly,

$$\lim_{x \to 1^-}\left(x^2\right) = 1$$

One-sided limits are not often used with continuous functions such as this one because it is simpler just to use a general limit:

$$\lim_{x \to 1}\left(x^2\right) = 1$$

In piece functions, however, one-sided limits are needed, as in the next example.

Sample

Let g be defined as follows: $g(x) = \begin{cases} \sin x \text{ for } x \geq 0 \\ \cos \dfrac{x}{2} \text{ for } x < 0 \end{cases}$

Find $\lim\limits_{x \to 0^+} g(x)$ and $\lim\limits_{x \to 0^-} g(x)$.

Begin by graphing the function.

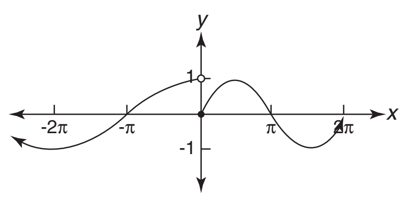

Figure 2.11

For the limit from the right

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (\sin x)$$
$$= 0$$

And for the limit from the left

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \left(\cos \frac{x}{2}\right)$$
$$= 1$$

As mentioned before, if the one-sided limits are not equal, then the general limit does not exist.

$$\left.\begin{array}{l} \lim_{x \to 0^+} g(x) = 0 \\ \lim_{x \to 0^-} g(x) = 1 \end{array}\right\} \Rightarrow \lim_{x \to 0} g(x) \text{ does not exist}$$

In general, if $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then $\lim_{x \to a} f(x)$ does not exist.

This method provides one method that is commonly used on free-response questions to justify the answer that a limit does not exist.

Functions with implied domain restrictions are a second case where one-sided limits are required.

Sample

What is $\lim_{x \to 2^+} \sqrt{x-2}$?

As a result of the domain of $y = \sqrt{x-2}$ a one-sided limit must be used for this problem. Because x can only be greater than or equal to 2, it would be impossible to approach 2 from the left. The function is continuous on its domain, however, so just substitute:

$$\lim_{x \to 2^+} \sqrt{x-2} = \sqrt{2-2} = 0$$

A graph may also be useful. The graph for this problem is the top half of a parabola.

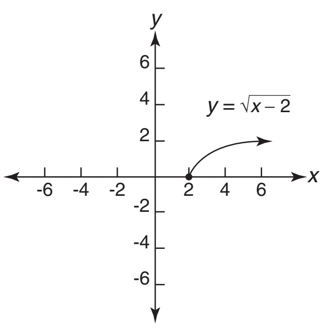

Figure 2.12

$$y = \sqrt{x-2} \Leftrightarrow y^2 = x - 2 \text{ where } y \geq 0$$
$$x = (y-0)^2 + 2 \text{ where } y \geq 0$$
$$\text{vertex}\,(2, 0), \text{opens right}$$

Infinite Limits

Another type of nonexistent limit that can require the use of one-sided limits is the **infinite limit** mentioned previously. The formal definition of an infinite limit is not on the AP exam.

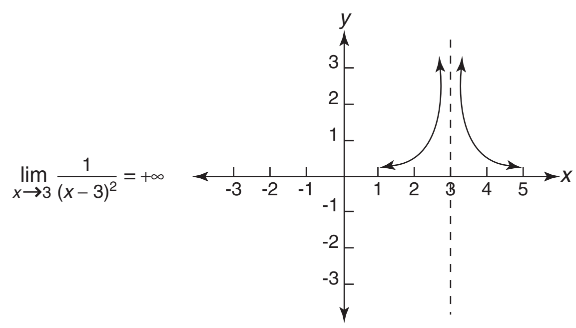

Figure 2.13

There are essentially two ways to find infinite limits:

1. Graph the function and look for vertical asymptotes.
2. Apply the theorem for infinite limits (given below).

Graphing the function is effective when the function is not terribly complicated. For example, with simple rational functions, locating the vertical asymptotes requires only finding any points where the denominator equals zero. For the foregoing function,

$$f(x) = \frac{1}{(x-3)^2} \Rightarrow \text{vertical asymptote where } (x-3)^2 = 0$$

Thus there is a vertical asymptote where $x = 3$, so

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty$$

Some functions may not even have to be graphed, but only visualized. Examples include

$$\lim_{x \to (\pi/2)^+} (\tan x) = -\infty \text{ and } \lim_{x \to (\pi/2)^-} (\tan x) = +\infty$$

Note the application of one-sided limits here. See the section on trig functions if no visualization suggests itself.

Graphing can be impractical, so sometimes it may be easier to apply the following theorem.

Theorem for Infinite Limits

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty \text{ if } \begin{cases} \lim_{x \to a} f(x) = \text{any constant} \\ \quad\quad \textit{and} \\ \lim_{x \to a} g(x) = 0 \end{cases}$$

The validity of this theorem can be demonstrated by examining a set of fractions where the numerator is held constant while the denominator approaches 0:

$$\frac{2}{4}, \frac{2}{3}, \frac{2}{2}, \frac{2}{1}, \frac{2}{\frac{1}{2}}, \frac{2}{\frac{1}{3}}, \frac{2}{\frac{1}{4}}, \frac{2}{\frac{1}{10}}, \frac{2}{\frac{1}{100}}, \frac{2}{\frac{1}{1000}}, \text{etc.}$$

As the denominator becomes arbitrarily small, the fractions are increasing without bound; that is, the limit is infinite.

Sample

Use the theorem for infinite limits to show that

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty$$

(from the original example in this section).

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = +\infty \text{ because } \begin{cases} \lim_{x \to 3} 1 = 1 \text{ (a constant)} \\ \quad\quad \textit{and} \\ \lim_{x \to 3} (x-3)^2 = 0 \end{cases}$$

A simpler notation that may be helpful is

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = \left(\frac{1}{0}\right) = +\infty$$

Note that (1/0) is enclosed in parentheses. (1/0) is, of course, an undefined expression; it is used here as a convenient notation for infinite limits. Determining the sign of the answer, $+\infty$ or $-\infty$, requires three steps:

1. Find the sign of the numerator by substitution.
2. Find how the denominator approaches zero, from negative numbers or positive numbers, by choosing a value from the correct side (if it's a one-sided limit) and substituting.
3. Apply the usual division rules to your results from the first two steps.

Sample

Find $\lim_{x \to 2^+} \frac{x-3}{x-2}$ by applying the theorem for infinite limits.

Because the limit of the numerator is a constant and the limit of the denominator is 0, the limit of the quotient is $\pm\infty$.

$$\left.\begin{array}{r}\lim_{x \to 2^+}(x-3) = -1 \\ \lim_{x \to 2^+}(x-2) = 0\end{array}\right\} \Rightarrow \lim_{x \to 2^+} \frac{x-3}{x-2} = \pm\infty$$

But is it $+\infty$ or $-\infty$? Use the three steps listed above.

1. The sign of the numerator is negative: $\lim_{x \to 2^+}(x-3) = -1$
2. Choose $x = 3$, a number to the *right* of 2, because that is the indicated direction on the limit, and substitute this into the denominator. Because $3 - 2 = 1$, a positive number, the denominator approaches 0 from positive values.
3. Applying the division rule

   $$\frac{\text{negative}}{\text{positive}} = \text{negative}$$

   reveals that the limit is $-\infty$.

Modifying the simplified notation from above, the symbol 0^+ or 0^- can be used to indicate the behavior of the denominator:

$$\lim_{x \to 2^+} \frac{x-3}{x-2} = \left(\frac{-1}{0^+}\right) = -\infty$$

and for the other side of the limit:

$$\lim_{x \to 2^-} \frac{x-3}{x-2} = \left(\frac{-1}{0^-}\right) = +\infty$$

On a free-response problem, you may have to find the vertical asymptotes of a function and then use calculus to show, or "justify," that they are truly vertical asymptotes. If so, just show that the function increases or decreases without bound by demonstrating that the limit is $\pm\infty$:

$x = a$ is a vertical asymptote of the function $f(x) \Leftrightarrow \lim_{x \to a^{+\text{ or }-}} f(x) = +\infty$ or $-\infty$

Limits at Infinity

An interesting variation on the concept of a limit involves taking the limit as the *x*-coordinates increase or decrease without bound, rather than as the *x*-coordinates approach a specific value. The best way to interpret this is graphically:

$\lim_{x \to \infty} \frac{2x+1}{x+2} = 2$

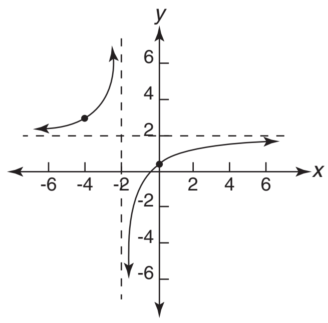

Figure 2.14

As the x-coordinates increase without bound ($x \to +\infty$), the y-coordinates (function values) get closer and closer to 2, which implies that the limit, or limiting value, of the function is 2. The function values never actually have to equal 2 for the limit to be 2, as long as they get closer and closer to 2, as shown in the following chart:

x	–1	0	1	2	3	4	10	100	1,000
$f(x)$	–1	0.5	1	1.25	1.4	1.5	1.75	1.97	1.997

From the graph, it should be obvious that as x decreases without bound ($x \to -\infty$), the function values approach 2. It should be obvious that horizontal asymptotes are thus also justified by limits.

$$y = a \text{ is a horizontal asymptote of the function } f(x) \Leftrightarrow \lim_{x \to \pm\infty} f(x) = a$$

As with infinite limits, there are two basic methods for finding limits at infinity:

1. Graph the function and look for horizontal asymptotes.
2. Use the theorem on page 84.

Graphing or visualizing the function in order to determine its horizontal asymptotes is probably the quickest and easiest method for finding limits at infinity. To find the horizontal asymptotes of a rational function, recall that you need to examine the degree of the numerator and denominator (see page 41) as shown in the next two examples.

Sample

What is $\lim_{x \to \infty}\left(\frac{3x^2 - 27}{8 - 2x^2}\right)$?

degree of numerator = degree of denominator

$\Rightarrow$ divide to find asymptote

$$
\begin{array}{r}
-\tfrac{3}{2} \\
-2x^2+8\overline{\smash{)}3x^2-27} \\
\underline{3x^2-12} \\
-15
\end{array}
\Rightarrow y=-\tfrac{3}{2} \text{ is the horizontal asymptote as shown on the graph}
$$

Therefore,

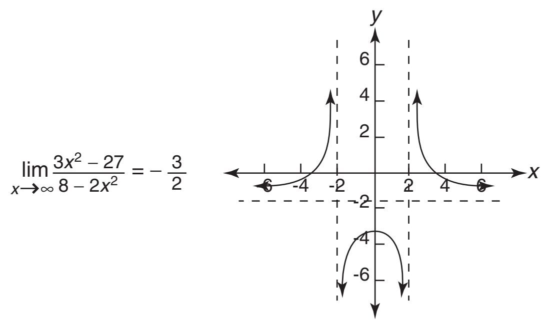

Figure 2.15

A shortcut: When the degree of the numerator equals the degree of the denominator, the limit is equal to the quotient of the leading coefficients of the numerator and denominator. In this example, the leading coefficient of the numerator is 3 and that of the denominator is –2, so

$$\lim_{x \to \infty} \frac{3x^2-27}{8-2x^2} = \frac{3}{-2} = -\frac{3}{2}$$

Sample

> What is $\lim_{x \to \infty} \frac{x-3}{4-2x^2}$?

degree of numerator < degree of denominator

$\Rightarrow$ divide to find asymptote

$$
\begin{array}{r}
0 \\
-2x^2+4\overline{\smash{)}x-3}
\end{array}
$$

$\Rightarrow$ $y = 0$ (the x-axis) is the horizontal asymptote

Thus

$$\lim_{x \to \infty} \frac{x-3}{4-2x^2} = 0$$

A shortcut: When the degree of the numerator is less than the degree of the denominator, the x-axis is the asymptote, so the limit is always 0.

Sample

What is $\lim_{x \to \infty}(\arctan x)$?

Some horizontal asymptotes should be known, and in these cases you may be able to avoid any graphing:

$$\lim_{x \to \infty}(\arctan x) = \frac{\pi}{2}$$

A graph will not suffice for justifying limits at infinity or horizontal asymptotes on a free-response problem. It is necessary to apply the following theorem in that case.

Theorem for Limits at Infinity

$$\lim_{x \to \pm\infty} \frac{a}{x^n} = 0 \text{ where } a = \text{any constant}$$

$$n = \text{any positive constant}$$

Here are some examples of the application of that theorem.

$$\lim_{x \to -\infty} \frac{2}{x^2} = 0 \qquad \lim_{x \to \infty} \frac{-4}{\sqrt{x}} = 0$$

An appealing reason for the validity of this theorem is found by examining a sequence of fractions where the numerator stays constant and the denominator increases without bound:

$$\frac{3}{\frac{1}{2}} \quad \frac{3}{1} \quad \frac{3}{2} \quad \frac{3}{4} \quad \frac{3}{6} \quad \frac{3}{10} \quad \frac{3}{100} \quad \frac{3}{1000} \quad \frac{3}{1,000,000}$$

This series is obviously approaching zero.

The theorem for limits at infinity is very useful when applied in conjunction with a simple rule: Rewrite the function in a new form by multiplying the numerator and denominator by the fraction $1/(x^h)$ where h is the highest power of any term in the function.

Sample

What is $\lim_{x \to \infty} \frac{3x - 2x^2}{7x^2 + 5}$?

The highest power in the function is 2, so multiply the numerator and denominator by $1/(x^2)$.

$$\lim_{x \to \infty} \frac{(3x - 2x^2)\left(\frac{1}{x^2}\right)}{(7x^2 + 5)\left(\frac{1}{x^2}\right)} = \lim_{x \to \infty}\left(\frac{\frac{3}{x} - 2}{7 + \frac{5}{x^2}}\right)$$

$$= \frac{0 - 2}{7 + 0} = \frac{-2}{7}$$

In the solution above, notice the theorem is used twice:

$$\lim_{x \to \infty} \frac{3}{x} = 0 \text{ and } \lim_{x \to \infty} \frac{5}{x^2} = 0$$

A sneaky version of limits at infinity often arises on the AP exam:

Sample

Find $\lim\limits_{x \to +\infty} \dfrac{\sqrt{x^2+2}}{2x-5}$ and $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2+2}}{2x-5}$.

For both of these problems, the highest power is actually first degree. But, in order to be able to simplify the expression, you must multiply the numerator and denominator by $1/\sqrt{x^2}$ rather than $1/x$.

$$\lim_{x \to +\infty} \left(\frac{\sqrt{x^2+2}}{2x-5} \right) \left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} \right) = \lim_{x \to +\infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{2}{x^2}}}{(2x-5)\frac{1}{\sqrt{x^2}}}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{2}{x^2}}}{(2x-5)\frac{1}{x}}$$

$$= \lim_{x \to +\infty} \frac{\sqrt{1 + \frac{2}{x^2}}}{\left(2 - \frac{5}{x}\right)} = \frac{\sqrt{1}}{2} = \frac{1}{2}$$

For the limit as x decreases without bound, all the previous work applies except for the simplification of the multiplier in the denominator (from the first to the second lines below). For the limit as x decreases without bound, x is a negative number, so $\sqrt{x^2} = -x$.

$$\lim_{x \to -\infty} \frac{\sqrt{x^2+2}}{2x-5} = \lim_{x \to -\infty} \left(\frac{\sqrt{x^2+2}}{2x-5} \right) \left(\frac{\frac{1}{\sqrt{x^2}}}{\frac{1}{\sqrt{x^2}}} \right)$$

$$= \lim_{x \to -\infty} \frac{\sqrt{\frac{x^2}{x^2} + \frac{2}{x^2}}}{(2x-5)\left(\frac{1}{-x}\right)}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{1 + \frac{2}{x^2}}}{\left(-2 + \frac{5}{x}\right)} = \frac{\sqrt{1}}{-2} = -\frac{1}{2}$$

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Special Trig Limits

Two special trig limits occur frequently on the AP exam, both in standard forms and with subtle variations. They are

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Both of these limits can be proved in a variety of ways, including L'Hôpital's rule, which is covered in the chapter on applications of the derivative. For now, memorize both limits, and learn how to apply them. Do not try direct substitution of 0 into either function; substituting yields the indeterminate form 0/0.

Sample

Find $\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta}$.

In order for you to apply the special trig limits directly, the arguments must match. This can be achieved by multiplying the numerator and denominator by 2.

$$\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta}\left(\frac{2}{2}\right) = \lim_{\theta \to 0}\left(\frac{\sin 2\theta}{2\theta}\right)2$$
$$= 1 \cdot 2 = 2$$

Sample

Find $\lim_{\alpha \to 0} \frac{\cos \alpha - 1}{\alpha}$.

$$\lim_{\alpha \to 0} \frac{\cos \alpha - 1}{\alpha} = \lim_{\alpha \to 0} -\left(\frac{1 - \cos \alpha}{\alpha}\right) = -1(0) = 0$$

Continuity

Continuity is an important calculus concept that is closely related to limits. An intuitive understanding of continuity is easy. If you can draw a function without having to lift your pencil, then the function is continuous. Conversely, if you have to lift your pencil for any reason, you have found a point of discontinuity. Study the graph of the following function and see if you can identify the points where it is discontinuous.

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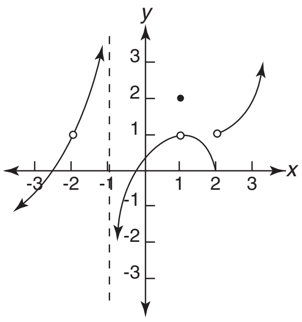

Figure 2.16

$f(x)$ is discontinuous at $x = -2, -1, 1,$ and 2.

In many cases, simply examining a graph may be sufficient to determine whether a function is continuous and to find any points of discontinuity. However, on free-response problems, proving or justifying continuity (or discontinuity) at a point may be required. You can do this by using the following three-part definition.

Definition of Continuity at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ if

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

All three conditions must be true for the function to be continuous at a.

Discontinuities such as those shown in Figure 2.16 at $x = -2$ (a hole) and $x = 1$ (a hole with an "extra" point) are called **removable discontinuities.** Those at $x = -1$ (an asymptote) and $x = 2$ (a break or skip) are called **nonremovable discontinuities.**

Sample

Use the definition of continuity to prove that $f(x)$ as graphed in Figure 2.16 is discontinuous at $x = -2, -1, 1,$ and 2.

In order to show that a function is discontinuous at a point, show that at least one of the three parts of the definition of continuity is not satisfied. It is not necessary (nor is it always possible) to show that all three parts of the definition are not satisfied.

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At $x = -2$	$f(-2)$ does not exist (hole)	discontinuous
At $x = -1$	$f(-1)$ does not exist (asymptote)	discontinuous
	or $\lim\limits_{x \to -1} f(x)$ does not exist	discontinuous
At $x = 1$	$\left.\begin{array}{l} f(1) = 2 \\ \lim\limits_{x \to 1} f(x) = 1 \end{array}\right\} \Rightarrow \lim\limits_{x \to 1} f(x) \neq f(1)$	discontinuous
At $x = 2$	$\left.\begin{array}{l} \lim\limits_{x \to 2^-} f(x) = 0 \\ \lim\limits_{x \to 2^+} f(x) = 1 \end{array}\right\} \Rightarrow \lim\limits_{x \to 2} f(x)$ does not exist	discontinuous

Continuity on an interval is defined by considering the interval to be a group of points and applying the definition of continuity at a point to each point in the interval.

Definition of Continuity on an Open Interval

$f(x)$ is continuous on the interval (a, b) if $f(x)$ is continuous for every point c where $c \in (a, b)$.

Definition of Continuity on a Closed Interval

For closed intervals, a slight modification of this definition is required in order to ensure continuity at the two endpoints. $f(x)$ is continuous on the interval $[a, b]$ if

1. $f(x)$ is continuous for every point c where $c \in (a, b)$
2. $\lim\limits_{x \to a^+} f(x) = f(a)$
3. $\lim\limits_{x \to b^-} f(x) = f(b)$

Justifying continuity often arises in the context of piece functions. One type of problem commonly found on the AP exam requires that you determine a value of a variable to guarantee continuity of a piece function and then justify the answer by applying the definition. This is illustrated in the next example.

Sample

Let f be defined as follows:

$$f(x) = \begin{cases} -x - 3 & \text{for } x \leq -2 \\ ax + b & \text{for } -2 < x < 1 \\ x^2 & \text{for } x \geq 1 \end{cases}$$

Find a and b such that the function is continuous. Justify your answer using the definition of continuity.

If this problem were to appear as a multiple-choice question, it would simply be a matter of forcing the line $y = ax + b$ to contain the points (–2, –1) and (1, 1).

$$(-2,-1)\text{ and }(1,1) \Rightarrow m = \frac{-1-1}{-2-1} = \frac{2}{3}$$

$$\text{point/slope form} \Rightarrow y-(-1) = \frac{2}{3}(x-(-2))$$

$$y = \frac{2}{3}x + \frac{1}{3}$$

Therefore, $a = \frac{2}{3}$ and $b = \frac{1}{3}$.

But if this is a free-response question, you must now justify this choice of a and b via the definition of continuity. $f(x)$ is now

$$f(x) = \begin{cases} -x-3 & \text{for } x \le -2 \\ \frac{2}{3}x + \frac{1}{3} & \text{for } -2 < x < 1 \\ x^2 & \text{for } x \ge 1 \end{cases}$$

Thus, at $x = -2$

1. $f(-2) = -1$, so $f(-2)$ exists
2. $$\left.\begin{array}{l} \lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} (-x-3) = -1 \\ \lim\limits_{x \to -2+} f(x) = \lim\limits_{x \to -2+} \left(\frac{2}{3}x + \frac{1}{3}\right) = -1 \end{array}\right\} \Rightarrow \lim_{x \to -2} f(x) = -1 \text{ so a limit exists}$$
3. $\lim\limits_{x \to -2} f(x) = f(-2) = -1$

Therefore, $f(x)$ is continuous at $x = -2$, since all three parts of the definition are satisfied.

Now, at $x = 1$,

1. $f(1) = 1$, so $f(1)$ exists
2. $$\left.\begin{array}{l} \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (\frac{2}{3}x + \frac{1}{3}) = 1 \\ \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x^2) = 1 \end{array}\right\} \Rightarrow \lim_{x \to 1} f(x) = 1 \text{ so a limit exists}$$
3. $\lim\limits_{x \to 1} f(x) = f(1) = 1$

Therefore, $f(x)$ is continous at $x = 1$.

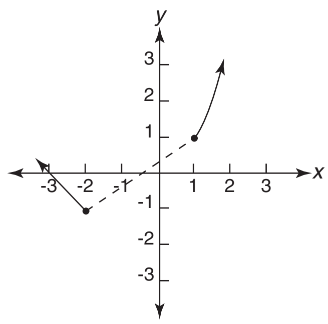


Figure 2.17

Sample Multiple-Choice Questions: Limits and Continuity

1. What is $\lim\limits_{x \to 2^+} \dfrac{x^3 - 8}{x - 2}$?

A. 0

B. 12

C. $+\infty$

D. $-\infty$

E. none of these

2. Let f be defined as follows:

$$f(x) = \begin{cases} 12 - 3x & \text{for } x \le 3 \\ 15 - x & \text{for } x > 3 \end{cases}$$

What is $\lim\limits_{x \to 3^+} f(x)$?

A. 3

B. 12

C. 15

D. 21

E. The limit does not exist.

3. What is $\lim\limits_{x \to 4} \sqrt[3]{\dfrac{7x}{x - 3}}$?

A. 1

B. $\sqrt[3]{7}$

C. 3

D. $\sqrt[3]{28}$

E. The limit does not exist.

4. Find $\lim\limits_{x \to 1/2} \|x\|$ (where $\| \;\|$ indicates the greatest-integer function).

A. 0

B. $\frac{1}{2}$

C. 1

D. 2

E. The limit does not exist.

5. What is $\lim\limits_{x \to \infty} \dfrac{3 - \sqrt{x}}{2\sqrt{x} + 5}$?

A. $-\frac{1}{2}$

B. $-\frac{1}{5}$

C. $\frac{3}{5}$

D. $\frac{3}{2}$

E. undefined

6. What is $\lim_{x \to \infty} \frac{2x-1}{3x+2}$?

A. $+\infty$

B. $-\infty$

C. $\frac{1}{3}$

D. $\frac{2}{3}$

E. 1

7. What is $\lim_{x \to \infty} \frac{4x^2}{x^2 + 10,000x}$?

A. 0

B. $\frac{1}{2500}$

C. 1

D. 4

E. The limit does not exist.

8. Which of the following statements is or are true?

I. $\lim_{x \to 2}(x^2 + 2x - 1) = 7$

II. $\lim_{x \to -3} \frac{x^2 + 5x + 6}{x^2 - x - 12} = \frac{1}{7}$

III. $\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x} = \pm\infty$

A. I and II only

B. I and III only

C. II and III only

D. III only

E. I, II, and III

9. What is $\lim_{x \to 2^-} \frac{5}{x-2}$?

A. 0

B. $-\frac{5}{4}$

C. $+\infty$

D. $-\infty$

E. none of these

10. What is $\lim_{x \to 2^+} \frac{2x}{x^2 - 4}$?

A. $\frac{-1}{2}$

B. 0

C. 2

D. $+\infty$

E. $-\infty$

11. What is $\lim_{x \to 0} \frac{\tan x}{2x}$?

A. 0

B. $\frac{1}{2}$

C. 1

D. 2

E. undefined

12. What is $\lim_{\phi \to 0} \frac{\cos\phi - 1}{\phi}$?

A. –1

B. 0

C. 1

D. ∞

E. none of these

13. Which of the following is NOT necessary to establish in order to show that a function $f(x)$ is continuous at the point $x = c$?

A. $f(c)$ exists

B. domain of $f(x)$ is all real numbers

C. $\lim_{x \to c} f(x) = f(c)$

D. $\lim_{x \to c} f(x)$ exists

E. All of these are necessary.

14. Let f be defined as follows:

$$f(x)=\begin{cases}\dfrac{x^2-1}{x-1} & \text{for } x\neq 1\\ 4 & \text{for } x=1\end{cases}$$

Which of the following statements is or are true?

I. $\lim\limits_{x\to 1} f(x)$ exists

II. $f(1)$ exists

III. $f(x)$ is continuous at $x = 1$

A. I only

B. II only

C. I and II only

D. I and III only

E. I, II, and III

15. Determine a value of k such that $f(x)$ is continuous, where

$$f(x)=\begin{cases}3kx-5 & \text{for } x>2\\ 4x-5k & \text{for } x\leq 2\end{cases}$$

A. 1

B. $\frac{13}{11}$

C. $\frac{3}{11}$

D. $\frac{-3}{11}$

E. -3

16. Find the value of k such that the following function is continuous for all real numbers.

$$f(x)=\begin{cases}kx-1 & \text{for } x<2\\ kx^2 & \text{for } x\geq 2\end{cases}$$

A. 1

B. $\frac{1}{2}$

C. $-\frac{1}{6}$

D. $-\frac{1}{2}$

E. none of these

17. The function $f(x)=\dfrac{x^2+5x+6}{x^2-4}$ has

A. only a removable discontinuity at $x = -2$

B. only a removable discontinuity at $x = 2$

C. a removable discontinuity at $x = -2$ and a nonremovable discontinuity at $x = 2$

D. removable discontinuities at $x = -2$ and $x = -3$

E. nonremovable discontinuities at $x = 2$ and $x = -3$

Answers to Multiple-Choice Questions

1. B. $\lim_{x \to 2^+} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2^+} \frac{(x-2)(x^2 + 2x + 4)}{x - 2}$

$= \lim_{x \to 2^+} (x^2 + 2x + 4) = 12$

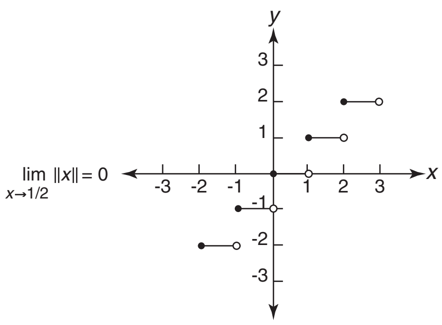
2. B. The problem asks for the limit from the right, so use the part of the piece function where, x > 3 — that is, $y = 15 - x$.

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (15 - x) = 12$$

3. D. $y = \sqrt[3]{\frac{7x}{x-3}}$ is discontinuous only at $x = 3$, so just substitute:

$\lim_{x \to 4} = \sqrt[3]{\frac{7x}{x-3}} = \sqrt[3]{\frac{7(4)}{4-3}} = \sqrt[3]{28}$

4. A. The greatest-integer function is $\| x \| : x \to$ greatest integer less than or equal to x. It looks like this:



5. A. $\lim_{x \to \infty} \frac{3 - \sqrt{x}}{2\sqrt{x} + 5} = \lim_{x \to \infty} \left(\frac{3 - \sqrt{x}}{2\sqrt{x} + 5} \right) \left(\frac{\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}} \right)$

$= \lim_{x \to \infty} \frac{\frac{3}{\sqrt{x}} - 1}{2 + \frac{5}{\sqrt{x}}} = \frac{0 - 1}{2 + 0} = \frac{-1}{2}$

6. D. $\lim_{x \to \infty} \frac{2x - 1}{3x + 2} = \lim_{x \to \infty} \left(\frac{2x - 1}{3x + 2} \right) \left(\frac{\frac{1}{x}}{\frac{1}{x}} \right)$

$= \lim_{x \to \infty} \frac{2 - \frac{1}{x}}{3 + \frac{2}{x}} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$

7. D. $\lim_{x \to \infty} \frac{4x^2}{x^2 + 10,000x} = \lim_{x \to \infty} \left(\frac{4x^2}{x^2 + 10,000x} \right) \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{4}{1 + \frac{10,000}{x}} = \frac{4}{1+0} = 4$

8. A. Examine each statement individually:

I. $\lim_{x \to 2} (x^2 + 2x - 1) = 2^2 + 2(2) - 1 = 7$, so I is true.

II. $\lim_{x \to -3} \frac{x^2 + 5x + 6}{x^2 - x - 12} = \lim_{x \to -3} \frac{(x+2)(x+3)}{(x-4)(x+3)}$

$= \lim_{x \to -3} \frac{x+2}{x-4} = \frac{-1}{-7} = \frac{1}{7}$, so II is true.

III. $\lim_{x \to 9} \frac{3 - \sqrt{x}}{9 - x} = \lim_{x \to 9} \left(\frac{3 - \sqrt{x}}{9 - x} \right) \left(\frac{3 + \sqrt{x}}{3 + \sqrt{x}} \right)$

$= \lim_{x \to 9} \frac{9 - x}{(9 - x)(3 + \sqrt{x})}$

$= \lim_{x \to 9} \frac{1}{3 + \sqrt{x}} = \frac{1}{6}$, so III is false.

9. D. $\lim_{x \to 2^-} \frac{5}{x - 2} = \left(\frac{5}{0^-} \right) = -\infty$

10. D. $\lim_{x \to 2^+} \frac{2x}{x^2 - 4} = \left(\frac{4}{0^+} \right) = +\infty$

11. B. $\lim_{x \to 0} \frac{\tan x}{2x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \right) \left(\frac{1}{2} \right) \left(\frac{1}{\cos x} \right)$

$= (1) \left(\frac{1}{2} \right) (1) = \frac{1}{2}$

12. B. $\lim_{\phi \to 0} \frac{\cos \phi - 1}{\phi} = \lim_{\phi \to 0} \frac{-1(1 - \cos \phi)}{\phi} = (-1)(0) = 0$

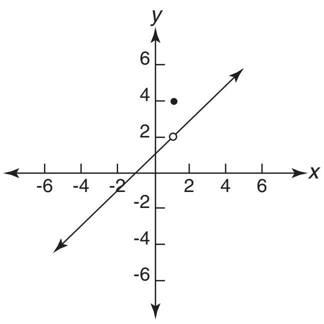
13. B. The definition of continuity at a point is made up of A, C, and D, so all three are necessary. Because the problem requires continuity only at a point, not on the set of real numbers, the domain does *not* need to be the real numbers.

14. C. The first part of the function can be simplified:

$$\frac{x^2-1}{x-1}=\frac{(x+1)(x-1)}{x-1}=x+1, x\neq 1$$

Now $f(x)=\begin{cases} x+1 & \text{for } x\neq 1 \\ 4 & \text{for } x=1 \end{cases}$

which is graphed below.



Statement I is true because $\lim_{x\to 1} f(x)=2$.

Statement II is also true, because the second part of the piece function yields $f(1)=4$.

Statement III is false. For $f(x)$ to be continuous at 1, $\lim_{x\to 1} f(x)$ must equal $f(1)$.

15. B. The function needs to "hook up" at $x=2$.

$f(x)=\begin{cases} 3kx-5 & \text{for } x>2 \\ 4x-5k & \text{for } x\leq 1 \end{cases}$

For the left piece: $f(2)=3k(2)-5=6k-5$
For the right piece: $f(2)=4(2)-5k=8-5k$

$\Rightarrow 6k-5=8-5k$ for continuity

$11k=13$

$k=\frac{13}{11}$

16. D. The function needs to "hook up" at $x = 2$.

$$f(x) = \begin{cases} kx - 1 & \text{for } x < 2 \\ kx^2 & \text{for } x \geq 2 \end{cases}$$

$$\left.\begin{array}{ll} \text{For the left piece:} & f(2) = k(2) - 1 = 2k - 1 \\ \text{For the right piece:} & f(2) = k(2)^2 = 4k \end{array}\right\}$$

$\Rightarrow 2k - 1 = 4k$ for continuity

$$-2k = 1$$

$$k = \frac{-1}{2}$$

17. C. $f(x) = \dfrac{x^2 + 5x + 6}{x^2 - 4} = \dfrac{(x+2)(x+3)}{(x+2)(x-2)} = \dfrac{x+3}{x-2}, x \neq -2$

$f(x)$ is a rational function with a hole at $\left(-2, \frac{-1}{4}\right)$ and an asymptote at $x = 2$. Therefore, there is a removable discontinuity at $x = -2$ and a nonremovable discontinuity at $x = 2$.

Sample Free-Response Questions: Limits and Continuity

1. Let f be defined as follows:

$$f(x) \begin{cases} 2x + 1 & \text{for } x \leq -2 \\ ax^2 + b & \text{for } -2 < x < 1 \\ \ln x & \text{for } x \geq 1 \end{cases}$$

Find values for a and b such that the function is continuous and use the definition of continuity to justify your answer.

2. Consider the function $f(x) = \dfrac{x^2 - 2x}{x^3 - 9x}$ on the interval [–5, 5].

A. Give any zeros of $f(x)$.

B. Give equations of *all* asymptotes. Justify your answer.

C. List all points where $f(x)$ is discontinuous. Use the definition of continuity to justify your answer.

D. Sketch $f(x)$.

Answers to Free-Response Questions

1. First, find a and b. The function must contain the points (–2, –3) and (1, 0) because these are the "ends" of the known pieces. A quick sketch may help.

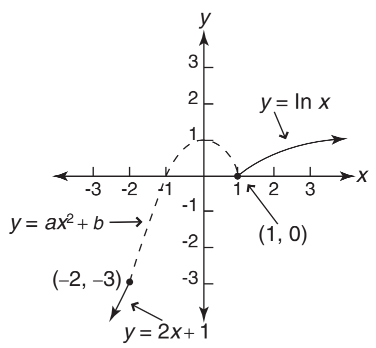
$$(-2,-3) \text{ on } y = ax^2 + b \Rightarrow -3 = 4a + b$$

$$(1,0) \text{ on } y = ax^2 + b \Rightarrow \underline{0 = a + b}$$

$$-3 = 3a$$

$$-1 = a \quad \Rightarrow b = 1, \text{ so } y = -x^2 + 1 \text{ for continuity}$$

$$\text{Thus } f(x) = \begin{cases} 2x+1 & \text{for } x \le 2 \\ -x^2+1 & \text{for } -2 < x < 1 \\ \ln x & \text{for } x \ge 1 \end{cases}$$



Now show that these values for a and b imply continuity by using the definition.

Justification by using the definition of continuity:

$f(x)$ is continuous at $x = a$ if:

1. $f(a)$ exists,

2. $\lim\limits_{x \to a} f(x)$ exists, and

3. $\lim\limits_{x \to a} f(x) = f(a)$

At $x = -2$: $f(-2) = -3$

$$\left.\begin{aligned} \lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} (2x+1) = -3 \\ \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} (-x^2+1) = -3 \end{aligned}\right\} \Rightarrow \lim_{x \to -2} f(x) = -3$$

$$\lim_{x \to -2} f(x) = f(-2) = -3$$

Therefore, $f(x)$ is continuous by definition at $x = -2$ for $a = -1$ and $b = 1$.

At $x = 1$: $f(1) = 0$

$$\left.\begin{array}{l}\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (\ln x) = 0 \\ \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (-x^2 + 1) = 0\end{array}\right\} \Rightarrow \lim_{x \to 1} f(x) = 0$$

$$\lim_{x \to 1} f(x) = f(1) = 0$$

Therefore, $f(x)$ is continuous by definition at $x = 1$ for $a = -1$ and $b = 1$.

2. Always begin by simplifying.

$$f(x) = \frac{x^2 - 2x}{x^3 - 9x} = \frac{x(x-2)}{x(x^2-9)} = \frac{x-2}{x^2-9} \quad x \neq 0$$

Therefore, $f(x) = \dfrac{x-2}{(x+3)(x-3)} \quad x \neq 0$

(a) For zeros, set the numerator equal to zero.

$x - 2 = 0$

$x = 2$, so $(2, 0)$ is the only zero

(b) For vertical asymptotes, set the denominator equal to zero.

$(x + 3)(x - 3) = 0$

So $x = -3$ and $x = 3$ are the vertical asymptotes.

To justify the vertical asymptotes, show $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or $\lim\limits_{x \to a^-} f(x) = \pm\infty$.

For $x = -3$: $\quad \lim\limits_{x \to -3^+} \dfrac{x-2}{x^2-9} = \left(\dfrac{-5}{0^-}\right) = +\infty$

For $x = 3$: $\quad \lim\limits_{x \to 3^-} \dfrac{x-2}{x^2-9} = \left(\dfrac{1}{0^-}\right) = -\infty$

For the horizontal asymptote, recall that

degree of numerator < degree of denominator

$\Rightarrow x-$axis is asymptote

Therefore, $y = 0$ is the horizontal asymptote.

To justify the horizontal asymptote, show $\lim_{x \to +\infty} f(x) = a$ or $\lim_{x \to -\infty} f(x) = a$.

$$\lim_{x \to +\infty} \frac{x-2}{x^2-9} = \lim_{x \to +\infty} \left(\frac{x-2}{x^2-9}\right)\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$$

$$= \lim_{x \to +\infty} \frac{\frac{1}{x} - \frac{2}{x^2}}{1 - \frac{9}{x^2}} = \frac{0-0}{1-0} = 0$$

(c) $f(x)$ is discontinuous at $x = -3$, 0, and 3.

Justification by using the definition of continuity:

$f(x)$ is continuous at $x = a$ if:

1. $f(a)$ exists,

2. $\lim_{x \to a} f(x)$ exists, and

3. $\lim_{x \to a} f(x) = f(a)$

At $x = -3$:

$f(-3)$ does not exist
or
$\lim_{x \to -3} f(x)$ does not exist,
because there is a vertical
asymptote at $x = -3$
$\Rightarrow$ discontinuous at $x = -3$

At $x = 0$:

$f(0)$ does not exist $\Rightarrow$ discontinuous at $x = 0$

At $x = 3$:

$f(3)$ does not exist
or
$\lim_{x \to 3} f(x)$ does not exist,
because there is a vertical
asymptote at $x = 3$
$\Rightarrow$ discontinuous at $x = 3$

(d) Sketch. *Don't forget the hole at* $\left(0, \frac{2}{9}\right)$ *that results from the cancellation.*