1

Displacements, Strain, Stress and Energy

1.1 The Reference State

Continuum mechanics deals with the change of field variables due to external actions. Examples of field variables are displacements, stresses, temperatures and magnetic induction. Actions include mechanical forces, heating, and so on. In general, a reference state is chosen with respect to which the change of field variables is measured. Let the fields of interest be defined in the reference state in a set of points, the so-called material points, occupying a volume V_0 with a surface A_0 in Eucledian space $\mathbb{R}^3$ (Figure 1.1). Assume that the reference space is described by a set of curvilinear coordinates $\left\{X^K\right\}_{K=1,2,3}$ related to a rectangular system $\{Z^K\}_{K=1,2,3}$ by

$$Z^K = Z^K(X^1, X^2, X^3). \tag{1.1}$$

Coordinates in the reference state are also called *material coordinates*. Consider an infinitesimal vector $\mathrm{d}\boldsymbol{X}$. One can write

$$\mathrm{d}\boldsymbol{X} = \frac{\partial \boldsymbol{X}}{\partial Z^K}\,\mathrm{d}Z^K \tag{1.2}$$

(summation over repeated indices).

$$\boldsymbol{I}_K = \frac{\partial \boldsymbol{X}}{\partial Z^K} \tag{1.3}$$

is a set of basis vectors in the rectangular system. Accordingly, $\boldsymbol{I}_K, K = 1, 2, 3$ do not depend on Z^K. In an analogous way, one can write

$$\mathrm{d}\boldsymbol{X} = \frac{\partial \boldsymbol{X}}{\partial X^K}\,\mathrm{d}X^K. \tag{1.4}$$

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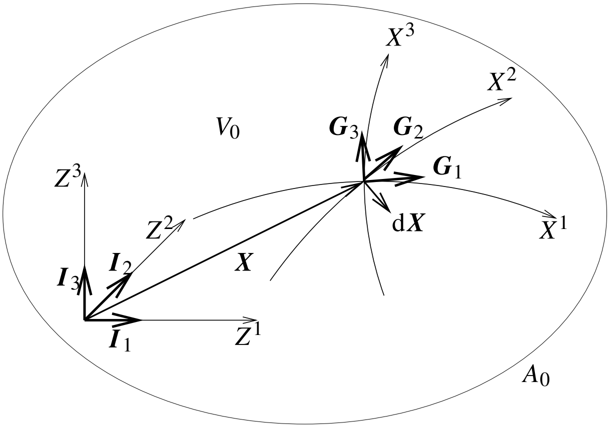

Figure 1.1 Material coordinate systems

The vectors

$$G_K = \partial X/\partial X^K \tag{1.5}$$

constitute a basis in the curvilinear coordinate system. One can write (compare Equation (1.2) with Equation (1.4))

$$G_K \, dX^K = I_L \, dZ^L \tag{1.6}$$

or

$$G_K = \frac{\partial Z^L}{\partial X^K} I_L. \tag{1.7}$$

The size dS of a vector dX is defined as

$$dS^2 := dX \cdot dX \tag{1.8}$$

where the "·" denotes the inner product of two vectors (also called the *dot product* or the *contraction of two vectors*). In rectangular coordinates, one finds (substitute Equation (1.2) into Equation (1.8))

$$\begin{aligned} dS^2 &= I_K \, dZ^K \cdot I_L \, dZ^L \\ &= dZ^K \, dZ^L I_K \cdot I_L \\ &=: dZ^K \, dZ^L I_{KL}. \end{aligned} \tag{1.9}$$

The metric tensor I_{KL} takes the value 1 for $K = L$ and 0 for $K \neq L$. In curvilinear coordinates, one obtains (substitute Equation (1.4) into Equation (1.8)),

$$\begin{aligned} dS^2 &= G_K \, dX^K \cdot G_L \, dX^L \\ &= dX^K \, dX^L G_K \cdot G_L \\ &=: dX^K \, dX^L G_{KL} \end{aligned} \tag{1.10}$$

G_{KL} is called the *metric tensor* for the coordinate system $\{X^K\}$. In general, $G_{KL} \neq 0$ for $K \neq L$, and $G_{KL} \neq 1$ for $K = L$. Thus, the basis vectors $\boldsymbol{G}_K$ are not necessarily orthonormal. Using the set $\{\boldsymbol{G}_K\}$, one can define another set $\{\boldsymbol{G}^L\}$ through the relations

$$\boldsymbol{G}_K \cdot \boldsymbol{G}^L = \delta_K{}^L \tag{1.11}$$

where $\delta_K{}^L = 0$ for $K \neq 0$ and $\delta_K{}^L = 1$ for $K = L$. In modern Riemannian geometry, $\{\boldsymbol{G}^L\}$ are called *one-forms* (or *covariant tensors of rank 1*). They map the vectors $\{\boldsymbol{G}_K\}$ (which are also called *contravariant tensors of rank 1*) into a scalar by Equation (1.11). $\{\boldsymbol{G}^L\}$ forms a basis for the vector space of one-forms and is also called the *dual basis of* $\{\boldsymbol{G}_k\}$. If $\boldsymbol{\alpha}$ is a one-form, one writes

$$\boldsymbol{\alpha} = \alpha_K \boldsymbol{G}^L. \tag{1.12}$$

The dot product of a vector $\boldsymbol{V}$ and a one-form $\boldsymbol{\alpha}$ is defined by

$$\boldsymbol{V} \cdot \boldsymbol{\alpha} = V^K \boldsymbol{G}_K \cdot \alpha_L \boldsymbol{G}^L = V^K \alpha_K \tag{1.13}$$

through Equation (1.11). In the same way, the dot product of two vectors and two one-forms yields

$$\boldsymbol{V} \cdot \boldsymbol{W} = V^K \boldsymbol{G}_K \cdot W^L \boldsymbol{G}_L = V^K W^L G_{KL} \tag{1.14}$$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \alpha_K \boldsymbol{G}^K \cdot \beta_L \boldsymbol{G}^L = \alpha_K \beta_L G^{KL} \tag{1.15}$$

where G^{KL} is defined by

$$G^{KL} := \boldsymbol{G}^K \cdot \boldsymbol{G}^L. \tag{1.16}$$

Notice that in Equations (1.13), (1.14) and (1.15) the same symbol is used for the dot product. The context shows whether a (covariant or contravariant) metric tensor is needed. Multiplying a vector $\boldsymbol{V}$ with the one-form $\boldsymbol{G}^L$ yields

$$\boldsymbol{V} \cdot \boldsymbol{G}^L = V^K \boldsymbol{G}_K \cdot \boldsymbol{G}^L = V^K \delta_K{}^L = V^L. \tag{1.17}$$

Thus, the components V^L of $\boldsymbol{V}$ can be obtained by taking the scalar product of $\boldsymbol{V}$ with the basis one-form $\boldsymbol{G}^L$. Hence,

$$\boldsymbol{V} = (\boldsymbol{V} \cdot \boldsymbol{G}^L)\boldsymbol{G}_L. \tag{1.18}$$

Similar statements to Equation (1.17) and Equation (1.18) can be made on the basis of one-forms:

$$\boldsymbol{\alpha} \cdot \boldsymbol{G}_L = \alpha_L \tag{1.19}$$

$$\boldsymbol{\alpha} = (\boldsymbol{\alpha} \cdot \boldsymbol{G}_L)\boldsymbol{G}^L. \tag{1.20}$$

Although the separation of tensors of rank one into vectors and one-forms is instructive from a theoretical point of view, there is no reason why a vector cannot be written in terms of a contravariant basis or a one-form in terms of a covariant basis. Substituting $\boldsymbol{G}^K$ in Equation (1.18) and $\boldsymbol{G}_K$ in Equation (1.19), one obtains

$$\boldsymbol{G}^K = G^{KL}\boldsymbol{G}_L \tag{1.21}$$

$$\boldsymbol{G}_K = G_{KL}\boldsymbol{G}^L. \tag{1.22}$$

The operation in Equation (1.21) and in Equation (1.22) is called *raising* and *lowering* of the index respectively. As we will see later on, some fields are naturally represented by covariant tensors (such as the Lagrangian strain and normals on a plane), whereas others are predestinate for a contravariant representation (such as stresses and normals in a direction). They can be viewed as dual fields.

1.2 The Spatial State

Because of the actions, the body B is mapped from its reference state into some other state, a spatial state. Let the spatial state be described by rectangular coordinates $\{z^k\}$ and curvilinear coordinates $\{x^k\}$. These coordinates are called *spatial coordinates*. The same definitions of the reference state apply to the spatial state, for instance,

$$\mathrm{d}s^2 = \mathrm{d}x^k\,\mathrm{d}x^l g_{kl} \tag{1.23}$$

where g_{kl} is the metric tensor of the spatial state. Within the theory of continuum mechanics, one tries to predict the spatial state from the reference state and the actions on it. Since

$$\boldsymbol{g}_k = \frac{\partial \boldsymbol{x}}{\partial x^k} \tag{1.24}$$

one can write

$$\begin{aligned}\mathrm{d}\boldsymbol{x} &= \mathrm{d}x^k \boldsymbol{g}_k = \frac{\partial x^k}{\partial X^K}\,\mathrm{d}X^K \boldsymbol{g}_k\\ &= x^k_{,K}\,\mathrm{d}X^K \boldsymbol{g}_k.\end{aligned} \tag{1.25}$$

This reveals that the spatial state can be predicted from the material state if $x^k_{,K}$ is known.

Defining the dyadic product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, written as $\boldsymbol{a} \otimes \boldsymbol{b}$ such that

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}) \tag{1.26}$$

and

$$\boldsymbol{c} \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b} \tag{1.27}$$

for an arbitrary vector $\boldsymbol{c}$, and similar for two one-forms or a vector and a one-form, one finds that

$$\begin{aligned}\mathrm{d}\boldsymbol{x} &= x^k_{,K}\boldsymbol{g}_k(\boldsymbol{G}^K \cdot \mathrm{d}\boldsymbol{X})\\ &= x^k_{,K}(\boldsymbol{g}_k \otimes \boldsymbol{G}^K) \cdot \mathrm{d}\boldsymbol{X}.\end{aligned} \tag{1.28}$$

Defining the deformation gradient $\boldsymbol{F}$ as

$$\boldsymbol{F} = x^k_{,K}(\boldsymbol{g}_k \otimes \boldsymbol{G}^K) \tag{1.29}$$

Equation (1.28) is transformed into

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}. \tag{1.30}$$

This shows that the deformation gradient is the Jacobian matrix of the motion from the material into the spatial state.

The dyadic product of two vectors, of two one-forms and of a vector and a one-form is called a *contravariant tensor of rank two*, a *covariant tensor of rank two* and a *mixed-variant tensor of rank two* respectively. If

$$\boldsymbol{a} = a_{KL}\boldsymbol{G}^K \otimes \boldsymbol{G}^L \tag{1.31}$$

then one can also write (Equation (1.21))

$$\boldsymbol{a} = a_{KL}G^{KM}G^{LN}\boldsymbol{G}_M \otimes \boldsymbol{G}_N = a^{MN}\boldsymbol{G}_M \otimes \boldsymbol{G}_N \tag{1.32}$$

where $a^{MN} = a_{KL}G^{KM}G^{LN}$ is obtained by raising the indices. Equation (1.31) is the covariant expansion of $\boldsymbol{a}$, Equation (1.32) is the contravariant expansion. To emphasize this, the notation $\boldsymbol{a}^\flat$ will be used for the covariant expansion and $\boldsymbol{a}^\sharp$ for the contravariant one (this agrees with recent literature, see (Marsden and Hughes 1983), (Holzapfel 2000)). Accordingly,

$$\boldsymbol{a}^\flat = a_{KL}\boldsymbol{G}^K \otimes \boldsymbol{G}^L \tag{1.33}$$

$$\boldsymbol{a}^\sharp = a^{KL}\boldsymbol{G}_K \otimes \boldsymbol{G}_L. \tag{1.34}$$

$\boldsymbol{F}$ is called a *mixed-variant two-point tensor* since it is the dyadic product of basis vectors belonging to different states (the material and the spatial state).

Notice that the dot on the right-hand side and on the left-hand side of Equation (1.26) have a different meaning: the dot on the right-hand side denotes the contraction of two vectors already encountered in Equation (1.8). The dot on the left-hand side symbolizes the contraction of a tensor of rank two and a vector. Whereas the contraction of two vectors is commutative, the contraction of a tensor of rank two and a vector is not

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}) \neq (\boldsymbol{c} \cdot \boldsymbol{a})\boldsymbol{b} = \boldsymbol{c} \cdot (\boldsymbol{a} \otimes \boldsymbol{b}). \tag{1.35}$$

However,

$$(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}) = (\boldsymbol{c} \cdot \boldsymbol{b})\boldsymbol{a} = \boldsymbol{c} \cdot (\boldsymbol{b} \otimes \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} \tag{1.36}$$

where

$$(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} := \boldsymbol{b} \otimes \boldsymbol{a} \tag{1.37}$$

is the transpose of $\boldsymbol{a} \otimes \boldsymbol{b}$.

The length ds of a vector d$\boldsymbol{x}$ in the spatial state satisfies

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} \\ &= x^k_{,K}\, \mathrm{d}X^K \boldsymbol{g}_k \cdot x^l_{,L}\, \mathrm{d}X^L \boldsymbol{g}_l \\ &= x^k_{,K} x^l_{,L}\, \mathrm{d}X^K\, \mathrm{d}X^L \boldsymbol{g}_k \cdot \boldsymbol{g}_l \\ &= x^k_{,K} x^l_{,L} g_{kl}\, \mathrm{d}X^K\, \mathrm{d}X^L. \end{aligned} \tag{1.38}$$

Defining the right Cauchy–Green tensor by

$$\boldsymbol{C} := C_{KL} \boldsymbol{G}^K \otimes \boldsymbol{G}^L \tag{1.39}$$

where

$$C_{KL} = x^k_{,K} x^l_{,L} g_{kl} \tag{1.40}$$

one obtains

$$\mathrm{d}s^2 = C_{KL}\, \mathrm{d}X^K\, \mathrm{d}X^L. \tag{1.41}$$

Comparing Equation (1.23) and Equation (1.41), one notices that for the calculations of ds^2, the tensor $\boldsymbol{C}$ is the equivalent of $\boldsymbol{g}$ in the reference frame. One also says that $\boldsymbol{C}$ is the pullback of $\boldsymbol{g}$ and, equivalently, $\boldsymbol{g}$ is the push-forward of $\boldsymbol{C}$. Equation (1.41) also shows that the Cauchy–Green tensor is positive definite. Furthermore, it satisfies

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} \tag{1.42}$$

where $\boldsymbol{F}^{\mathrm{T}}$ is the transpose of $\boldsymbol{F}$ defined by

$$\boldsymbol{F}^{\mathrm{T}} := x^k_{,K} (\boldsymbol{G}^K \otimes \boldsymbol{g}_k). \tag{1.43}$$

Indeed, since $(\boldsymbol{a} \otimes \boldsymbol{b}) \cdot (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{a} \otimes \boldsymbol{d}) \boldsymbol{b} \cdot \boldsymbol{c}$, one finds

$$\begin{aligned} \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} &= x^k_{,K} x^l_{,L} (\boldsymbol{G}^K \otimes \boldsymbol{g}_k) \cdot (\boldsymbol{g}_l \otimes \boldsymbol{G}^L) \\ &= x^k_{,K} x^l_{,L} g_{kl} \boldsymbol{G}^K \otimes \boldsymbol{G}^L. \end{aligned} \tag{1.44}$$

The stretch in a direction $\boldsymbol{N} = (\mathrm{d}X^K/\mathrm{d}S)\boldsymbol{G}_K$ is defined by

$$\begin{aligned} \lambda_{(N)} = \frac{\mathrm{d}s}{\mathrm{d}S} &= \sqrt{C_{KL} \frac{\mathrm{d}X^K}{\mathrm{d}S} \frac{\mathrm{d}X^L}{\mathrm{d}S}} \\ &= \sqrt{C_{KL} N^K N^L} \end{aligned} \tag{1.45}$$

where $N^K = \mathrm{d}X^K/\mathrm{d}S$. Thus, $\lambda_{(N)}$ is the change of length of an infinitesimal vector in direction $\boldsymbol{N}$ in the reference state.

If the mapping $\boldsymbol{x}(\boldsymbol{X})$ is one to one, it can be inverted to yield $\boldsymbol{X}(\boldsymbol{x})$. Since matter cannot disappear, the Jacobian determinant

$$J := \det(x^k_{,K}) = \det \boldsymbol{F} \tag{1.46}$$

cannot be zero and the mapping is one to one. Assuming the transformation to be continuous, this means that J must be either everywhere positive or everywhere negative. Since $J = 1$ for the identical transformation, it is everywhere positive.

$\mathrm{d}S^2$ can also be written as

$$\begin{aligned}\mathrm{d}S^2 = \mathrm{d}X^K\,\mathrm{d}X^L G_{KL} &= X^K_{,k} X^L_{,l}\,\mathrm{d}x^k\,\mathrm{d}x^l G_{KL} \\ &= (b^{-1})_{kl}\,\mathrm{d}x^k\,\mathrm{d}x^l \qquad (1.47)\end{aligned}$$

where

$$(b^{-1})_{kl} := X^K_{,k} X^L_{,l} G_{KL}. \qquad (1.48)$$

The tensor $\boldsymbol{b}$ (the inverse of $\boldsymbol{b}^{-1}$) is called the *left Cauchy–Green tensor* and satisfies

$$b^{kl} = x^k_{,K} x^l_{,L} G^{KL} \qquad (1.49)$$

or, equivalently,

$$\boldsymbol{b} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{T}}. \qquad (1.50)$$

Consequently,

$$\boldsymbol{b}^{-1} = \boldsymbol{F}^{-\mathrm{T}} \cdot \boldsymbol{F}^{-1} \qquad (1.51)$$

where

$$\boldsymbol{F}^{-1} = X^K_{,k} \boldsymbol{G}_K \otimes \boldsymbol{g}^k \qquad (1.52)$$

is the inverse of the deformation gradient and

$$\boldsymbol{F}^{-\mathrm{T}} = X^K_{,k} \boldsymbol{g}^k \otimes \boldsymbol{G}_K. \qquad (1.53)$$

Equation (1.47) shows that, with respect to $\mathrm{d}S^2$, $\boldsymbol{b}^{-1}$ plays in the spatial state the role that is assumed by $\boldsymbol{G}$ in the reference state, that is,

$$G_{KL}\,\mathrm{d}X^K\,\mathrm{d}X^L = (b^{-1})_{kl}\,\mathrm{d}x^k\,\mathrm{d}x^l. \qquad (1.54)$$

Therefore, $\boldsymbol{b}^{-1}$ is called the *push-forward* of $\boldsymbol{G}$ and equivalently $\boldsymbol{G}$ is called the *pullback* of $\boldsymbol{b}^{-1}$. Equation (1.41) and Equation (1.47) can also be written as

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{X} \cdot \boldsymbol{C} \cdot \mathrm{d}\boldsymbol{X} \qquad (1.55)$$

and

$$\mathrm{d}S^2 = \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{b}^{-1} \cdot \mathrm{d}\boldsymbol{x}. \qquad (1.56)$$

Since J is the determinant of $x^k_{,K}$, one also has

$$\frac{\partial J}{\partial x^k_{,K}} = \mathrm{cofactor}(x^k_{,K}). \qquad (1.57)$$

The cofactor of $x^k_{,K}$ is defined as the *determinant of the matrix one obtains after deleting row k and column K in* $x^k_{,K}$ *(this is the so-called minor determinant of* $x^k_{,K}$*), multiplied by* $(-1)^{k+K}$. Equation (1.57) is easily derived by recalling that the determinant of a matrix can be obtained by taking the dot product of any row with the row of the corresponding cofactors, for example, if the first row is used,

$$J = x^1_{,1}\text{cofactor }(x^1_{,1}) + x^1_{,2}\text{cofactor }(x^1_{,2}) + x^1_{,3}\text{cofactor }(x^3_{,3}). \tag{1.58}$$

$X^K_{,k}$ is the inverse of $x^k_{,K}$. Accordingly,

$$X^K_{,k} = \frac{1}{J}\text{cofactor }(x^k_{,K}). \tag{1.59}$$

Indeed, the inverse of a matrix $\boldsymbol{M}$ satisfies (Greenberg 1978)

$$(M^{-1})^{KL} = \frac{1}{\det \boldsymbol{M}}\text{cofactor }(M^{LK}). \tag{1.60}$$

Comparing Equation (1.57) with Equation (1.59), one finds

$$\frac{\partial J}{\partial x^k_{,K}} = J X^K_{,k}. \tag{1.61}$$

This relationship will be needed for the time derivative of J.

So far, only length changes were considered. Since the determinant of a map describes its volume change, one can write

$$\mathrm{d}v = J\,\mathrm{d}V \tag{1.62}$$

where $\mathrm{d}v$ and $\mathrm{d}v$ are infinitesimal volume elements in the reference and spatial configuration respectively. Denoting an infinitesimal surface element in the reference configuration by the one-form $\mathrm{d}\boldsymbol{A}$ orthogonal to the surface element and with size equal to the area of the surface, and similarly for the spatial configuration, one obtains for Equation (1.62),

$$\mathrm{d}\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{x} = J\,\mathrm{d}\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{X} \tag{1.63}$$

or

$$\mathrm{d}\boldsymbol{a} \cdot \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} = J\,\mathrm{d}\boldsymbol{A} \cdot \mathrm{d}\boldsymbol{X}. \tag{1.64}$$

Since this applies to an arbitrary vector $\mathrm{d}\boldsymbol{X}$, one finds

$$\mathrm{d}\boldsymbol{a} = J\,\mathrm{d}\boldsymbol{A} \cdot \boldsymbol{F}^{-1} \tag{1.65}$$

or

$$\mathrm{d}\boldsymbol{a} = J\boldsymbol{F}^{-\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A}. \tag{1.66}$$

This is feasible since it expresses that for isochoric (volume-preserving, $J = 1$) motion, the surface change is inversely proportional to the length change.

1.3 Strain Measures

Physically, we are interested in the change from d$\boldsymbol{X}$ to d$\boldsymbol{x}$ and not as much in the actual size of d$\boldsymbol{x}$. After all, assuming the body to be stress-free at the outset of the calculation, it is the change of d$\boldsymbol{X}$ that generates the stress field in a mechanical problem. The vector connecting the initial position of a material particle at $\boldsymbol{X}$ to its new position $\boldsymbol{x}$ at time t is called the *displacement* $\boldsymbol{U}(\boldsymbol{X}, t)$ of that particle at time t. One can write (see Figure 1.2)

$$\boldsymbol{u} = \boldsymbol{U} = \boldsymbol{o} + \boldsymbol{x} - \boldsymbol{X} \tag{1.67}$$

$\boldsymbol{o}$ is the vector connecting the spatial frame of reference $\{\boldsymbol{g}_k\}$ with the material frame $\{\boldsymbol{G}_K\}$. Since the displacement connects a material vector with a spatial vector, it does not uniquely belong to the material nor to the spatial frame, and both upper case notation $\boldsymbol{U}$ and lower case notation $\boldsymbol{u}$ will be used. The component notation yields

$$\boldsymbol{U} = U^K \boldsymbol{G}_K \tag{1.68}$$

and

$$\boldsymbol{u} = u^k \boldsymbol{g}_k. \tag{1.69}$$

The difference between dS^2 and ds^2 can be written as (Equation (1.41))

$$\mathrm{d}s^2 - \mathrm{d}S^2 = (C_{KL} - G_{KL})\,\mathrm{d}X^K\,\mathrm{d}X^L \tag{1.70}$$

as well as (Equation (1.47))

$$\mathrm{d}s^2 - \mathrm{d}S^2 = (g_{kl} - b_{kl}^{-1})\,\mathrm{d}x^k\,\mathrm{d}x^l. \tag{1.71}$$

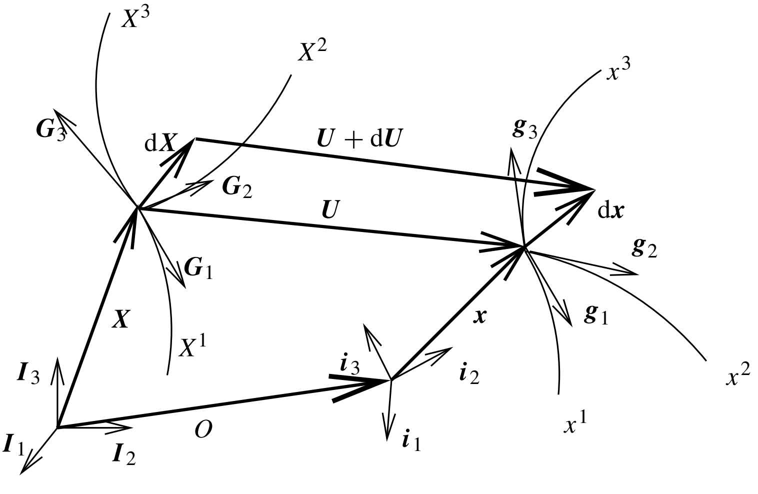

Figure 1.2 Displacement vectors

Now, the Lagrangian strain tensor $\boldsymbol{E}$ (also sometimes called the *Green–Lagrange strain tensor*) is defined by

$$E_{KL} := \tfrac{1}{2}(C_{KL} - G_{KL}) \tag{1.72}$$

and the Eulerian strain tensor $\boldsymbol{e}$ (also sometimes called the *Euler–Almansi strain tensor*) by

$$e_{kl} := \tfrac{1}{2}(g_{kl} - b^{-1}_{kl}). \tag{1.73}$$

Accordingly,

$$\mathrm{d}s^2 - \mathrm{d}S^2 = 2E_{KL}\,\mathrm{d}X^K\,\mathrm{d}X^L \tag{1.74}$$

$$= 2e_{kl}\,\mathrm{d}x^k\,\mathrm{d}x^l. \tag{1.75}$$

$\boldsymbol{E}$ and $\boldsymbol{e}$ are second-order tensors and can be interpreted as measures for the change of length in a body. Using Equation (1.67) one can write

$$\begin{aligned}
\mathrm{d}s^2 - \mathrm{d}S^2 &= \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} - \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}\\
&= (\mathrm{d}\boldsymbol{U} + \mathrm{d}\boldsymbol{X})\cdot(\mathrm{d}\boldsymbol{U} + \mathrm{d}\boldsymbol{X}) - \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{X}\\
&= \mathrm{d}\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{X} + \mathrm{d}\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{U} + \mathrm{d}\boldsymbol{U}\cdot\mathrm{d}\boldsymbol{U}\\
&= (\boldsymbol{U}_{,K}\cdot\boldsymbol{X}_{,L} + \boldsymbol{X}_{,K}\cdot\boldsymbol{U}_{,L} + \boldsymbol{U}_{,K}\cdot\boldsymbol{U}_{,L})\,\mathrm{d}X^K\,\mathrm{d}X^L.
\end{aligned} \tag{1.76}$$

Since $\boldsymbol{U} = U^M\boldsymbol{G}_M$ and $\mathrm{d}\boldsymbol{X} = \mathrm{d}X^N\boldsymbol{G}_N$, one finds

$$\begin{aligned}
\frac{\partial\boldsymbol{U}}{\partial X^K} &= \frac{\partial}{\partial X^K}(U^M\boldsymbol{G}_M) = \frac{\partial U^M}{\partial X^K}\boldsymbol{G}_M + U^M\frac{\partial\boldsymbol{G}_M}{\partial X^K}\\
&= \frac{\partial U^M}{\partial X^K}\boldsymbol{G}_M + U^M\frac{\partial^2 Z^L}{\partial X^K\partial X^M}\boldsymbol{I}_L\\
&= \frac{\partial U^M}{\partial X^K}\boldsymbol{G}_M + U^M\frac{\partial^2 Z^L}{\partial X^K\partial X^M}\frac{\partial X^N}{\partial Z^L}\boldsymbol{G}_N\\
&= \left(\frac{\partial U^M}{\partial X^K} + U^N\frac{\partial^2 Z^L}{\partial X^K\partial X^N}\frac{\partial X^M}{\partial Z^L}\right)\boldsymbol{G}_M\\
&=: U^M_{;K}\boldsymbol{G}_M
\end{aligned} \tag{1.77}$$

and

$$\frac{\partial\boldsymbol{X}}{\partial X^L} = \boldsymbol{G}_L. \tag{1.78}$$

$U^M_{;K}$ is the covariant derivative of U^M and can also be written as

$$U^M_{;K} = U^M_{,K} + U^N\begin{Bmatrix} M\\ KN\end{Bmatrix} \tag{1.79}$$

where

$$\begin{Bmatrix} M \\ KN \end{Bmatrix} := \frac{\partial^2 Z^L}{\partial X^K \partial X^N} \frac{\partial X^M}{\partial Z^L} \tag{1.80}$$

are called the *Christoffel symbols* of the second kind. Hence,

$$\mathrm{d}s^2 - \mathrm{d}S^2 = (U^M_{;K} G_{LM} + U^M_{;L} G_{KM} + U^M_{;K} U^N_{;L} G_{MN})\, \mathrm{d}X^K\, \mathrm{d}X^L. \tag{1.81}$$

Comparison of Equation (1.74) with Equation (1.81) finally yields

$$2E_{KL} = U^M_{;K} G_{LM} + U^M_{;L} G_{KM} + U^M_{;K} U^N_{;L} G_{MN}. \tag{1.82}$$

Similarly, one finds

$$2e_{kl} = u^m{}_{;k} g_{lm} - u^m{}_{;l} g_{km} + u^m{}_{;k} u^n{}_{;l} g_{mn}. \tag{1.83}$$

It is important to note that the extra term in Equation (1.77) derives from the fact that $\boldsymbol{G}_M$ is not necessarily constant in space. The expression $U^M_{;K}$ is also called the *covariant derivative* covariant derivative of $\boldsymbol{U}$ (Eringen 1980). For rectangular coordinates, the unit vectors do not vary in space and Equations (1.82) and (1.83) reduce to

$$2E_{KL} = U^M_{,K} G_{LM} + U^M_{,L} G_{KM} + U^M_{,K} U^N_{,L} G_{MN} \tag{1.84}$$

and

$$2e_{kl} = u^m{}_{,k} g_{lm} - u^m{}_{,l} g_{km} + u^m{}_{,k} u^n{}_{,l} g_{mn}. \tag{1.85}$$

Furthermore, the distinction between $\{\boldsymbol{G}^K\}$ and $\{\boldsymbol{G}_K\}$ fades since both bases are identical, and G_{KL} is the unit tensor. Consequently, Equations (1.84) and (1.85) can be further simplified to

$$2E_{KL} = U_{K,L} + U_{L,K} + U_{M,K} U_{M,L} \tag{1.86}$$

and

$$2e_{kl} = u_{k,l} + u_{l,k} - u_{m,k} u_{m,l}. \tag{1.87}$$

The above equations establish a relationship between displacements and strains. This relationship is nonlinear owing to the last terms in Equations (1.82) to (1.87). In problems with small deformations, the nonlinear terms are frequently neglected, leading to the linear strain $\tilde{E}_{KL}$, in rectangular coordinates:

$$\tilde{E}_{KL} := \tfrac{1}{2}(U_{K,L} + U_{L,K}). \tag{1.88}$$

Defining the infinitesimal rotation as

$$\tilde{R}_{KL} := \tfrac{1}{2}(U_{K,L} - U_{L,K}) \tag{1.89}$$

Equation (1.86) can be rewritten as

$$E_{KL} = \tilde{E}_{KL} + \tfrac{1}{2}(\tilde{E}_{MK} + \tilde{R}_{MK})(\tilde{E}_{ML} + \tilde{R}_{ML}) \tag{1.90}$$

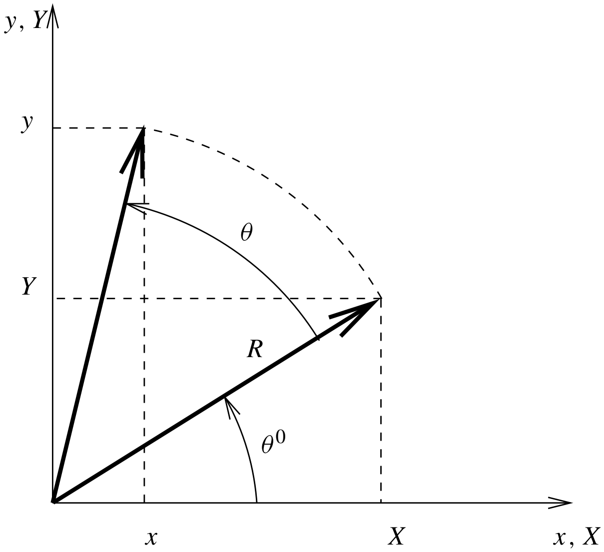

Figure 1.3 Finite rotation of a rod

showing that for the linear strain to be a good approximation for the actual strain, both the linear strain and the linear rotation must be small. Accordingly, for a rod freely rotating about one of its ends, linear strains are a poor approximation of the real strains. This is easily shown. Consider a rod of length R rotating about the origin (Figure 1.3). The original position is

$$\begin{aligned} X &= R\cos\theta^0 \\ Y &= R\sin\theta^0. \end{aligned} \tag{1.91}$$

The final position is characterized by

$$\begin{aligned} x &= R\cos(\theta^0 + \theta) \\ y &= R\sin(\theta^0 + \theta). \end{aligned} \tag{1.92}$$

Consequently, the displacements amount to

$$\begin{aligned} U_X &= x - X = X(\cos\theta - 1) - Y\sin\theta \\ U_Y &= y - Y = X\sin\theta + Y(\cos\theta - 1). \end{aligned} \tag{1.93}$$

The infinitesimal strains yield

$$\begin{aligned} \tilde{E}_{XX} &= U_{X,X} = \cos\theta - 1 \\ \tilde{E}_{YY} &= V_{Y,Y} = \cos\theta - 1 \\ \tilde{E}_{XY} &= (U_{X,Y} + U_{Y,X})/2 = 0 \end{aligned} \tag{1.94}$$

which shows that $\tilde{E}_{XX}$ and $\tilde{E}_{YY}$ are generally not zero. Since a rigid body motion must not generate strains, this clearly shows that the infinitesimal strains are not suited for finite rotations. The Lagrangian strain tensor, on the other hand, vanishes. For instance,

$$\begin{aligned} E_{XX} &= U_{X,X} + (U_{X,X}^2 + U_{Y,X}^2)/2 \\ &= (\cos\theta - 1) + \left[(\cos\theta - 1)^2 + \sin^2\theta\right]\Big/2 = 0. \end{aligned} \tag{1.95}$$

This is especially important for slender structures such as shells and beams in which strains are usually small but rotations can be large.

1.4 Principal Strains

An infinitesimal vector $\mathrm{d}\boldsymbol{X}$ with size $\mathrm{d}S$ is transformed by the motion of the body into $\mathrm{d}\boldsymbol{x}$ with size $\mathrm{d}s$ satisfying

$$\mathrm{d}s^2 - \mathrm{d}S^2 = E_{KL}\,\mathrm{d}X^K\,\mathrm{d}X^L \tag{1.96}$$

or

$$\frac{\mathrm{d}s^2 - \mathrm{d}S^2}{\mathrm{d}S^2} = E_{KL}N^K N^L \tag{1.97}$$

where

$$N^K := \frac{\mathrm{d}X^K}{\mathrm{d}S} \tag{1.98}$$

is a unit vector satisfying

$$N^K N^L G_{KL} = \frac{\mathrm{d}X^K \mathrm{d}X^L G_{KL}}{\mathrm{d}S^2} = 1. \tag{1.99}$$

The expression in Equation (1.97) is a measure for the relative change of length of a fiber originally parallel to $\boldsymbol{N}$. The question we want to look into now is the following: in which directions is this change of length maximal? This boils down to maximizing Equation (1.97) subject to the constraint Equation (1.99). The variables are the components of $\boldsymbol{N}$. Following the usual procedure of calculus, finding an extremum reduces to setting the derivative of the target function with respect to the variables to zero. The target function is

$$F(\boldsymbol{N}) = E_{KL}N^K N^L - \Lambda_E(N^K N^L G_{KL} - 1) \tag{1.100}$$

where Λ_E is a Lagrange multiplier. Hence,

$$\begin{gathered} \frac{\partial}{\partial N^M}[E_{KL}N^K N^L - \Lambda_E(N^K N^L G_{KL} - 1)] = 0 \\ \Updownarrow \\ E_{ML}N^L + E_{KM}N^K - \Lambda_E N^L G_{ML} - \Lambda_E N^K G_{KM} = 0 \\ \Updownarrow \\ (E_{KM} - \Lambda_E G_{KM})N^K = 0. \end{gathered} \tag{1.101}$$

This is a classical eigenvalue problem (generalized for curvilinear coordinates). Since $\boldsymbol{E}$ is a symmetric tensor, the eigenvalues Λ_E are real and the corresponding eigenvectors are mutually orthogonal (or can be made orthogonal). Indeed, suppose that Λ_E is a complex eigenvalue, then

$$E_{KM}N^K = \Lambda_E G_{KM} N^K. \tag{1.102}$$

Premultiplying with the complex conjugate of $\boldsymbol{N}$ yields

$$\overline{N}^M E_{KM} N^K = \Lambda_E \overline{N}^M G_{KL} N^K. \tag{1.103}$$

Since $\boldsymbol{E}$ is symmetric and real, one obtains

$$\begin{aligned}\overline{\overline{N}^M E_{KM} N^K} &= N^M E_{KM} \overline{N}^K = N^K E_{MK} \overline{N}^M \\ &= N^K E_{KM} \overline{N}^M\end{aligned} \tag{1.104}$$

and similar for $\boldsymbol{G}$. Consequently, $\overline{N}^M E_{KM} N^K$ and $\overline{N}^M G_{KM} N^K$ are real and Λ_E must be real because of Equation (1.103) and the positive-definiteness of $\boldsymbol{G}$. Because of Equation (1.102), the eigenvectors $\boldsymbol{N}$ are real too.

To prove that the eigenvectors are mutually orthogonal, consider two distinct eigenvalues Λ_1 and Λ_2 with two corresponding eigenvectors $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$. Then

$$E_{KM} N_1^K = \Lambda_1 G_{KM} N_1^K \tag{1.105}$$

and

$$E_{KM} N_2^K = \Lambda_2 G_{KM} N_2^K. \tag{1.106}$$

Multiplying Equation (1.105) with N_2^M and Equation (1.106) with N_1^M and subtracting both yields

$$N_2^M E_{KM} N_1^K - N_1^M E_{KM} N_2^K = \Lambda_1 N_2^M G_{KM} N_1^K - \Lambda_2 N_2^M G_{MK} N_1^K \tag{1.107}$$

or

$$N_2^M (E_{KM} - E_{MK}) N_1^K = \Lambda_1 N_2^M G_{KM} N_1^K - \Lambda_2 N_1^M G_{KL} N_2^K. \tag{1.108}$$

Since both $\boldsymbol{E}$ and $\boldsymbol{G}$ are symmetric, this yields

$$0 = (\Lambda_1 - \Lambda_2) N_2^M G_{KM} N_1^K. \tag{1.109}$$

Λ_1 and Λ_2 are assumed to be distinct, which means

$$N_1^K G_{KM} N_2^M = 0 \tag{1.110}$$

or

$$\boldsymbol{N}_1 \cdot \boldsymbol{N}_2 = 0. \tag{1.111}$$

This completes the proof.

The eigenvalues are the solution of a third-order nonlinear equation expressing that the determinant of the matrix in Equation (1.101) has to satisfy

$$\det(E_{KM} - \Lambda_E G_{KM}) = 0 \Leftrightarrow \det(E^K{}_M - \Lambda_E \delta^K{}_M) = 0 \tag{1.112}$$

for the equation to have nontrivial solutions. Since the extremal strains have a physical relevance and are independent of the coordinate system, the coefficients of Equation (1.112) are invariants. Indeed, Equation (1.112) can be written as

$$-\Lambda_E^3 + I_{1E}\Lambda_E^2 - I_{2E}\Lambda_E + I_{3E} = 0 \tag{1.113}$$

where

$$I_{1E} = \delta^K{}_L E^L{}_K = \mathrm{tr}\boldsymbol{E} \tag{1.114}$$

$$I_{2E} = \frac{1}{2}\left[I_{1E}^2 - \mathrm{tr}(\boldsymbol{E}^2)\right] \tag{1.115}$$

$$I_{3E} = \det \boldsymbol{E} = \frac{1}{3!} e_{LMP} e^{KNQ} E^L{}_K E^M{}_N E^P{}_Q \tag{1.116}$$

are the first, second and third invariant of $\boldsymbol{E}$. The expression $\mathrm{tr}\boldsymbol{E}$ stands for the trace of $\boldsymbol{E}$, e_{LMP} and e^{KNQ} are the alternating symbols: $e_{KLM} = 1$ for $KLM = 123$ or any cyclic rotation thereof, $e_{KLM} = -1$ for $KLM = 321$ or any cyclic rotation, else $e_{KLM} = 0$. The eigenvalues are called *principal strains* and the corresponding direction $\boldsymbol{N}_i$ are called *principal directions*. They are obtained by solving Equation (1.101) in which the solutions of Equation (1.113) are substituted. For the solution of Equation (1.113), which is a cubic equation, see (Simo and Hughes 1997) or (Abramowitz and Stegun 1972).

Since $\boldsymbol{E}$ and $\boldsymbol{C}$ differ by the metric tensor, Equation (1.101) can also be written as

$$(C_{KM} - \Lambda_C G_{KM})N^K = 0 \tag{1.117}$$

where

$$\Lambda_C = 2\Lambda_E + 1. \tag{1.118}$$

Consequently, the eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{E}$ are the same and the eigenvalues are directly related by Equation (1.118). In what follows, Λ_i denotes the eigenvalues of $\boldsymbol{C}$, that is, $\Lambda_i = \Lambda_{iC}$. The calculation of the eigenvectors $\boldsymbol{N}_i$ is somewhat tedious. Sometimes, it is more advantageous to calculate the tensors $\boldsymbol{N}^i \otimes \boldsymbol{N}^i$, which play the role of a tensorial basis. Here, $\boldsymbol{N}^i$ (index up) are the one-forms obtained by raising the index of $\boldsymbol{N}$:

$$\boldsymbol{N}^i = \boldsymbol{G}^\flat \cdot \boldsymbol{N}_i \tag{1.119}$$

and satisfy (Equation (1.110)

$$\boldsymbol{N}^i \cdot \boldsymbol{N}_j = \delta^i{}_j. \tag{1.120}$$

The one-forms $\{\boldsymbol{N}^i\}$ are the dual basis of the vectors $\{\boldsymbol{N}_i\}$.

Theorem 1.4.1 *Let* $\boldsymbol{C}$ *be a symmetric covariant second-order tensor in* R^3, Λ_i *its eigenvalues and* $\boldsymbol{N}_i$ *the corresponding eigenvectors, then*

$$\boldsymbol{C} = \sum_{i=1}^{3} \Lambda_i \boldsymbol{M}^i \tag{1.121}$$

where

$$\boldsymbol{M}^i = \boldsymbol{N}^{\underline{i}} \otimes \boldsymbol{N}^{\underline{i}} \tag{1.122}$$

and $\boldsymbol{N}^i$ *are the one-forms dual to* $\boldsymbol{N}_i$.

Proof.

$$\begin{aligned}
\sum_i \Lambda_i \boldsymbol{M}^i \cdot \boldsymbol{N}_l &= \sum_i \Lambda_i (\boldsymbol{N}^i \otimes \boldsymbol{N}^i) \cdot \boldsymbol{N}_l \\
&= \sum_i \Lambda_i \boldsymbol{N}^i (\boldsymbol{N}^i \cdot \boldsymbol{N}_l) \\
&= \sum_i \Lambda_i \boldsymbol{N}^i \delta^i{}_l \\
&= \Lambda_{\underline{l}} \boldsymbol{N}^{\underline{l}} = \Lambda_{\underline{l}} \boldsymbol{G}^\flat \cdot \boldsymbol{N}_{\underline{l}}, \forall l
\end{aligned} \tag{1.123}$$

where $\boldsymbol{G}^\flat$ is the covariant metric tensor and an underscore or a summation sign remove implicit summation. Consequently, $\boldsymbol{C}$ and $\sum_i \Lambda_i \boldsymbol{M}^i$ have the same eigenvalues and eigenvectors and are identical. Since $\boldsymbol{C}$ is a covariant tensor, it is logical that it is made up of one-forms and not of vectors.

An interesting property is

$$\begin{aligned}
\boldsymbol{C} \cdot \boldsymbol{C} &= \left(\sum_i \Lambda_i \boldsymbol{M}^i \right) \cdot \left(\sum_j \Lambda_j \boldsymbol{M}^j \right) \\
&= \sum_i \sum_j \Lambda_i \Lambda_j (\boldsymbol{N}^i \otimes \boldsymbol{N}^i) \cdot (\boldsymbol{N}^j \otimes \boldsymbol{N}^j) \\
&= \sum_i \sum_j \Lambda_i \Lambda_j (\boldsymbol{N}^i \otimes \boldsymbol{N}^j)(\boldsymbol{N}^i \cdot \boldsymbol{N}^j) \\
&= \sum_i \Lambda_i^2 (\boldsymbol{N}^i \otimes \boldsymbol{N}^i) = \sum_i \Lambda_i^2 \boldsymbol{M}^i .
\end{aligned} \tag{1.124}$$

This property allows for the following simple calculation of $\boldsymbol{M}_i$. Since

$$\left\{ \begin{array}{rcl}
\boldsymbol{M}^1 + \boldsymbol{M}^2 + \boldsymbol{M}^3 &=& \boldsymbol{G} \\
\Lambda_1 \boldsymbol{M}^1 + \Lambda_2 \boldsymbol{M}^2 + \Lambda_3 \boldsymbol{M}^3 &=& \boldsymbol{C} \\
\Lambda_1^2 \boldsymbol{M}^1 + \Lambda_2^2 \boldsymbol{M}^2 + \Lambda_3^2 \boldsymbol{M}^3 &=& \boldsymbol{C}^2
\end{array} \right. \tag{1.125}$$

one obtains

$$\boldsymbol{M}^i = \frac{1}{D} \left[\boldsymbol{C}^2 - (I_{1C} - \Lambda_i)\boldsymbol{C} + I_{3C} \Lambda_i^{-1} \boldsymbol{G} \right] \tag{1.126}$$

where

$$D_i = (\Lambda_{\underline{i}} - \Lambda_j)(\Lambda_{\underline{i}} - \Lambda_k) \tag{1.127}$$

for $j, k \neq i$.

If two eigenvalues are identical, for example, $\Lambda = \Lambda_1 = \Lambda_2 \neq \Lambda_3$ one obtains instead of Equation (1.125),

$$\left\{ \begin{array}{rcl} (\boldsymbol{M}^1 + \boldsymbol{M}^2) + \boldsymbol{M}^3 & = & \boldsymbol{G} \\ \Lambda(\boldsymbol{M}^1 + \boldsymbol{M}^2) + \Lambda_3 \boldsymbol{M}^3 & = & \boldsymbol{C} \\ \Lambda^2(\boldsymbol{M}^1 + \boldsymbol{M}^2) + \Lambda_3^2 \boldsymbol{M}^3 & = & \boldsymbol{C}^2 \end{array} \right. \tag{1.128}$$

Discarding the third equation, one finds

$$\boldsymbol{M}^1 + \boldsymbol{M}^2 = \frac{\Lambda_3 \boldsymbol{G} - \boldsymbol{C}}{\Lambda_3 - \Lambda} \tag{1.129}$$

$$\boldsymbol{M}^3 = \frac{\boldsymbol{C} - \Lambda \boldsymbol{G}}{\Lambda_3 - \Lambda}. \tag{1.130}$$

This means that $\boldsymbol{M}^1$ and $\boldsymbol{M}^2$ are not known individually, only their sum can be derived. For three equal eigenvalues, the set in Equation (1.125) reduces to the first equation (Itskov 2001). The tensors $\boldsymbol{M}^1$, $\boldsymbol{M}^2$ and $\boldsymbol{M}^3$ are sometimes called *structural tensors*. They are genuine tensors of rank two subject to Equation (1.122) and the normality condition of $\boldsymbol{N}^i$.

Notice that

$$\boldsymbol{C} \cdot \boldsymbol{M}_i = \boldsymbol{C} \cdot (\boldsymbol{N}_{\underline{i}} \otimes \boldsymbol{N}_{\underline{i}}) = (\boldsymbol{C} \cdot \boldsymbol{N}_{\underline{i}})\boldsymbol{N}_{\underline{i}} = \Lambda_{\underline{i}}(\boldsymbol{G} \cdot \boldsymbol{N}_{\underline{i}})\boldsymbol{N}_{\underline{i}} = \Lambda_{\underline{i}} \boldsymbol{G} \cdot \boldsymbol{M}_{\underline{i}} \tag{1.131}$$

and

$$\boldsymbol{C} : \boldsymbol{M}_i = \boldsymbol{C} : (\boldsymbol{N}_{\underline{i}} \otimes \boldsymbol{N}_{\underline{i}}) = \boldsymbol{N}_{\underline{i}} \cdot \boldsymbol{C} \cdot \boldsymbol{N}_{\underline{i}} = \boldsymbol{N}_{\underline{i}} \Lambda_{\underline{i}} \cdot (\boldsymbol{G} \cdot \boldsymbol{N}_{\underline{i}}) = \Lambda_i \tag{1.132}$$

since Equation (1.117) is equivalent to

$$\boldsymbol{C} \cdot \boldsymbol{N}_i = \Lambda_{\underline{i}} \boldsymbol{G} \cdot \boldsymbol{N}_{\underline{i}} \tag{1.133}$$

and the double contraction or inner product of two second-order tensors $\boldsymbol{a} \otimes \boldsymbol{b}$ and $\boldsymbol{c} \otimes \boldsymbol{d}$ is defined by

$$(\boldsymbol{a} \otimes \boldsymbol{b}) : (\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) = \operatorname{tr}\left[(\boldsymbol{a} \otimes \boldsymbol{b})^{\mathrm{T}} \cdot (\boldsymbol{c} \otimes \boldsymbol{d})\right]. \tag{1.134}$$

One finds that the eigenvalues λ_i of $\boldsymbol{F}$ satisfy

$$\lambda_i = \sqrt{\Lambda_i} \tag{1.135}$$

because of Equation (1.42). Defining

$$\boldsymbol{n}_i := \boldsymbol{F} \cdot \boldsymbol{N}_{\underline{i}} / \lambda_{\underline{i}} \tag{1.136}$$

one can write

$$\boldsymbol{F} = \sum_i \lambda_i (\boldsymbol{n}_i \otimes \boldsymbol{N}^i). \tag{1.137}$$

The normals $\boldsymbol{N}_i$ along the principal directions in the material frame are mapped into the normals $\boldsymbol{n}_i$ in the spatial frame, strained by an amount λ_i. Notice that Equation (1.136) actually defines the right-hand side of the eigenvalue problem for $\boldsymbol{F}$. Furthermore, since $\boldsymbol{F}$ is a two-point tensor, it cannot map a vector into a multiple of itself.

Not only are $\{\boldsymbol{N}_i\}$ mutually orthogonal but $\{\boldsymbol{n}_i\}$ are also a mutually orthogonal set of vectors. Indeed,

$$\begin{aligned} \boldsymbol{n}_i \cdot \boldsymbol{n}_j &= \frac{1}{\lambda_{\underline{i}}\lambda_{\underline{j}}} \boldsymbol{N}_{\underline{i}} \cdot \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} \cdot \boldsymbol{N}_{\underline{j}} \\ &= \frac{1}{\lambda_{\underline{i}}\lambda_{\underline{j}}} \boldsymbol{N}_{\underline{i}} \cdot \boldsymbol{C} \cdot \boldsymbol{N}_{\underline{j}} \\ &= \frac{\lambda_{\underline{j}}}{\lambda_{\underline{i}}} \boldsymbol{N}_{\underline{i}} \cdot \boldsymbol{N}_{\underline{j}} = \frac{\lambda_{\underline{j}}}{\lambda_{\underline{i}}} \delta_{\underline{i}\underline{j}}. \end{aligned} \tag{1.138}$$

Hence, in each material point, there exist three mutually orthogonal vectors, the deformation of which is extremal and yields again three mutually orthogonal vectors. The vectors $\{\boldsymbol{n}_i\}$ are the eigenvectors of the inverse of the left Cauchy–Green tensor $\boldsymbol{b}^{-1}$. Indeed, substituting Equation (1.136) into Equation (1.117) yields

$$\boldsymbol{C} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{n}_i = \Lambda_C \boldsymbol{G} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{n}_i. \tag{1.139}$$

Substituting $\boldsymbol{C}$ (Equation (1.42)) leads to

$$\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{g} \cdot \boldsymbol{n}_i = \Lambda_C \boldsymbol{G} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{n}_i \tag{1.140}$$

or

$$\frac{1}{\Lambda_C} \boldsymbol{g} \cdot \boldsymbol{n}_i = \boldsymbol{F}^{-\mathrm{T}} \cdot \boldsymbol{G} \cdot \boldsymbol{F}^{-1} \cdot \boldsymbol{n}_i \tag{1.141}$$

which is equivalent to

$$\frac{1}{\Lambda_C} \boldsymbol{g} \cdot \boldsymbol{n}_i = \boldsymbol{b}^{-1} \cdot \boldsymbol{n}_i. \tag{1.142}$$

At this point, the polar decomposition theorem should be mentioned because of its physical relevance. It states that the deformation gradient $\boldsymbol{F}$ can be written as the product of an orthogonal matrix $\boldsymbol{R}$ and a symmetric tensor $\boldsymbol{U}$, called the *right-stretch tensor*. Accordingly,

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} \tag{1.143}$$

where

$$\boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{-1} \tag{1.144}$$

and

$$\boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}}. \tag{1.145}$$

Since

$$\begin{aligned} \boldsymbol{C} &= \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} = \boldsymbol{U}^{\mathrm{T}} \cdot \boldsymbol{R}^{\mathrm{T}} \cdot \boldsymbol{R} \cdot \boldsymbol{U} \\ &= \boldsymbol{U}^{\mathrm{T}} \cdot \boldsymbol{U} = \boldsymbol{U} \cdot \boldsymbol{U} = \boldsymbol{U}^2 \end{aligned} \tag{1.146}$$

$\boldsymbol{U}$ and $\boldsymbol{F}$ have the same eigenvalues equal to the square root of the eigenvalues of $\boldsymbol{C}$. Since $\boldsymbol{C}$ is positive-definite (Equation (1.41)), $\boldsymbol{U}$ is also positive-definite. Furthermore, the eigenvectors of $\boldsymbol{C}$ and $\boldsymbol{U}$ are identical. We have

$$\boldsymbol{U} = \sum_i \lambda_i \boldsymbol{N}^i \otimes \boldsymbol{N}^i = \sum_i \sqrt{\Lambda_i} \boldsymbol{N}^i \otimes \boldsymbol{N}^i \tag{1.147}$$

and

$$\boldsymbol{R} = \sum_i \boldsymbol{n}_i \otimes \boldsymbol{N}_i. \tag{1.148}$$

Indeed,

$$\begin{aligned} \boldsymbol{R} \cdot \boldsymbol{U} &= \sum_i (\boldsymbol{n}_i \otimes \boldsymbol{N}_i) \sum_j \lambda_j (\boldsymbol{N}^j \otimes \boldsymbol{N}^j) \\ &= \sum_i \sum_j \lambda_j \boldsymbol{n}_i \otimes \boldsymbol{N}^j (\boldsymbol{N}_i \cdot \boldsymbol{N}^j) \\ &= \sum_i \lambda_i \boldsymbol{n}_i \otimes \boldsymbol{N}^i = \boldsymbol{F}. \end{aligned} \tag{1.149}$$

In a similar way, one can decompose $\boldsymbol{F}$ into

$$\boldsymbol{F} = \boldsymbol{V} \cdot \boldsymbol{R}. \tag{1.150}$$

$\boldsymbol{V}$ is the left-stretch tensor.

Equation (1.143) shows that the motion can be locally decomposed into a pure stretch along the principal directions followed by a rotation. It should be emphasized that a pure stretch is guaranteed for the principal directions only. For all other directions $\boldsymbol{N}$, the product $\boldsymbol{U} \cdot \boldsymbol{N}$ will involve some rotation, unless some of the principal values coincide. Furthermore, $\boldsymbol{R}$ is not constant in space. Consequently, $\boldsymbol{R}$ denotes a microscopic rotation in the material point of interest and not a macroscopic rotation.

1.5 Velocity

In most problems time is involved. The total time rate of change of a field is denoted by the total derivative D/Dt. It physically means that a material particle is followed while monitoring the change of some field at the momentaneous location of the moving particle. The partial derivative $\partial/\partial t$ is used when looking at the change in time of a field at a fixed spatial position. Both are related by (chain rule)

$$\frac{D}{Dt}\phi(\boldsymbol{x}, t) = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t} \tag{1.151}$$

where $\phi(\boldsymbol{x}, t)$ is some field variable. The second term in Equation (1.151) is also called the *convective time rate of change* and is solely due to the nonzero velocity of the particle. The vector field

$$\boldsymbol{v} := \frac{\partial \boldsymbol{x}(\boldsymbol{X}, t)}{\partial t} \tag{1.152}$$

is the classical velocity of a particle originally at location $\boldsymbol{X}$. Applying Equation (1.151) to the particle acceleration, $\boldsymbol{a}$ defined by

$$\boldsymbol{a} := \frac{D\boldsymbol{v}}{Dt}(\boldsymbol{X}, t) \tag{1.153}$$

one finds

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \cdot \frac{\partial \boldsymbol{x}}{\partial t} \tag{1.154}$$

$$= \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \otimes \nabla) \cdot \boldsymbol{v} \tag{1.155}$$

where

$$\nabla := \frac{\partial}{\partial \boldsymbol{x}} \tag{1.156}$$

is a one-form. Writing

$$\boldsymbol{v} = v^k \boldsymbol{g}_k \tag{1.157}$$

and

$$\nabla = \boldsymbol{g}^l \frac{\partial}{\partial x^l} \tag{1.158}$$

leads to

$$\begin{aligned} \nabla \otimes \boldsymbol{v} = (\boldsymbol{v} \otimes \nabla)^{\mathrm{T}} &= \boldsymbol{g}^l \otimes \frac{\partial}{\partial x^l}(v^k \boldsymbol{g}_k) \\ &= v^k_{;l} \boldsymbol{g}^l \otimes \boldsymbol{g}_k \end{aligned} \tag{1.159}$$

and, consequently,

$$\begin{aligned} a^k \boldsymbol{g}_k &= \frac{\partial v^k}{\partial t} \boldsymbol{g}_k + v^k_{;l} v^m (\boldsymbol{g}_k \otimes \boldsymbol{g}^l) \cdot \boldsymbol{g}_m \\ &= \left(\frac{\partial v^k}{\partial t} + v^k_{;l} v^l \right) \boldsymbol{g}_k \end{aligned} \tag{1.160}$$

or, in rectangular coordinates

$$a^k = \frac{\partial v^k}{\partial t} + v^k_{,l} v^l. \tag{1.161}$$

The change of length in time is given by

$$\begin{aligned}\frac{D}{Dt}\mathrm{d}s^2 &= \frac{D}{Dt}\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}\\ &= \left(\frac{D}{Dt}\mathrm{d}\boldsymbol{x}\right)\cdot\mathrm{d}\boldsymbol{x}+\mathrm{d}\boldsymbol{x}\cdot\left(\frac{D}{Dt}\mathrm{d}\boldsymbol{x}\right).\end{aligned}\tag{1.162}$$

Since

$$\frac{D}{Dt}\mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\cdot\mathrm{d}\boldsymbol{x} = (\boldsymbol{v}\otimes\nabla)\cdot\mathrm{d}\boldsymbol{x}\tag{1.163}$$

one finds

$$\begin{aligned}\frac{D}{Dt}\mathrm{d}s^2 &= \mathrm{d}\boldsymbol{x}\cdot(\nabla\otimes\boldsymbol{v})\cdot\mathrm{d}\boldsymbol{x}+\mathrm{d}\boldsymbol{x}\cdot(\boldsymbol{v}\otimes\nabla)\cdot\mathrm{d}\boldsymbol{x}\\ &= \mathrm{d}\boldsymbol{x}\cdot(\nabla\otimes\boldsymbol{v}+\boldsymbol{v}\otimes\nabla)\cdot\mathrm{d}\boldsymbol{x}\\ &= 2\mathrm{d}\boldsymbol{x}\cdot\boldsymbol{d}\cdot\mathrm{d}\boldsymbol{x}\end{aligned}\tag{1.164}$$

where

$$\boldsymbol{d} := \tfrac{1}{2}(\nabla\otimes\boldsymbol{v}+\boldsymbol{v}\otimes\nabla)^\flat\tag{1.165}$$

is called the *deformation rate tensor*. One also defines the velocity gradient $\boldsymbol{l}$ and the spin tensor $\boldsymbol{w}$:

$$\boldsymbol{l} := (\boldsymbol{v}\otimes\nabla)^\flat\tag{1.166}$$

$$\boldsymbol{w} := \tfrac{1}{2}(\boldsymbol{l}-\boldsymbol{l}^{\mathrm{T}}).\tag{1.167}$$

Consequently, one obtains

$$\boldsymbol{d} = \tfrac{1}{2}(\boldsymbol{l}+\boldsymbol{l}^{\mathrm{T}}).\tag{1.168}$$

Equation (1.164) shows that $2\boldsymbol{d}$ plays a similar role for $D(\mathrm{d}s^2)/Dt$ as $\boldsymbol{g}$ for $\mathrm{d}s^2$.

Since

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{X}\tag{1.169}$$

one finds by taking the total derivative of both sides

$$\begin{aligned}(\boldsymbol{v}\otimes\nabla)\cdot\mathrm{d}\boldsymbol{x} &= \dot{\boldsymbol{F}}\cdot\mathrm{d}\boldsymbol{X}\\ &= \dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1}\cdot\mathrm{d}\boldsymbol{x}\end{aligned}\tag{1.170}$$

or

$$\boldsymbol{l} = (\dot{\boldsymbol{F}}\cdot\boldsymbol{F}^{-1})^\flat\tag{1.171}$$

where

$$\dot{(\ )} := \overline{(\ )}^{\boldsymbol{\cdot}} := \frac{D}{Dt}(\ ). \tag{1.172}$$

In component notation, Equation (1.164) reads

$$\frac{D}{Dt}\,\mathrm{d}s^2 = 2\mathrm{d}x^k\,\mathrm{d}x^l\,d_{kl}. \tag{1.173}$$

Since

$$\mathrm{d}s^2 = (2E_{KL} + G_{KL})\,\mathrm{d}X^K\,\mathrm{d}X^L \tag{1.174}$$

one also finds

$$\frac{D}{Dt}\,\mathrm{d}s^2 = 2\dot{E}_{KL}\,\mathrm{d}X^K\,\mathrm{d}X^L. \tag{1.175}$$

Comparison of Equation (1.173) and Equation (1.175) leads to

$$\dot{E}_{KL} = d_{kl}x^k_{\ ,K}x^l_{\ ,L} \tag{1.176}$$

or

$$\dot{\boldsymbol{E}} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{d} \cdot \boldsymbol{F}. \tag{1.177}$$

Accordingly, the tensor $\dot{\boldsymbol{E}}$ is the pullback of $\boldsymbol{d}$ and equivalently $\boldsymbol{d}$ is the push-forward of $\dot{\boldsymbol{E}}$.

The time derivative of the Jacobian J can be derived as follows:

$$\begin{aligned}\frac{DJ}{Dt} &= \frac{DJ}{Dx^k_{\ ,K}}\frac{Dx^k_{\ ,K}}{Dt}\\ &= JX^K_{\ ,k}\left(\frac{Dx^k}{Dt}\right)_{,K}\\ &= JX^K_{\ ,k}v^k_{\ ,K}\\ &= JX^K_{\ ,k}v^k_{\ ;l}x^l_{\ ,K}\\ &= Jv^k_{\ ;k}.\end{aligned} \tag{1.178}$$

In this derivation, Equation (1.61) was used. The expression $v^k_{\ ;k}$ corresponds to the divergence of the velocity, also written as $\nabla \cdot \boldsymbol{v}$.

1.6 Objective Tensors

Observers are not always on the same place and they do not necessarily use the same time. Consequently, observations are made by people in totally different places characterized by local coordinate systems for time and space. In space, these coordinate systems are related by a translation described by a vector $\boldsymbol{c}(t)$ and a rotation defined by an orthogonal matrix

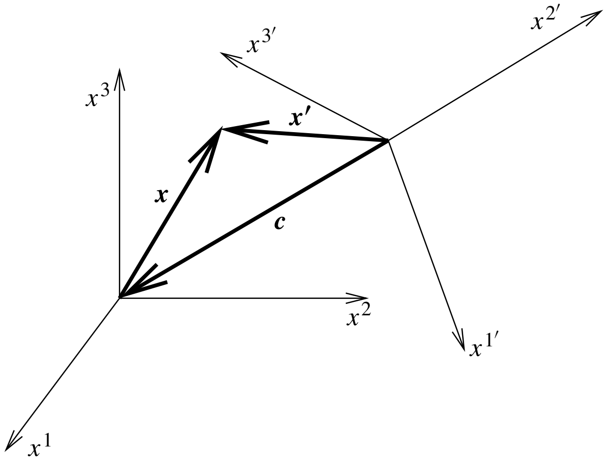

Figure 1.4 Frames of different observers

$Q(t)$ (Figure 1.4). Notice that, since the observers generally move with a different speed, c and Q are a function of the time t. The different wall-clock time can be expressed by a shift of time. Hence,

$$\boldsymbol{x}'(\boldsymbol{X}, t') = \boldsymbol{c}(t) + \boldsymbol{Q}(t) \cdot \boldsymbol{x}(\boldsymbol{X}, t) \tag{1.179}$$

$$t' = t - a. \tag{1.180}$$

Since $\boldsymbol{Q}$ is an orthogonal matrix $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{T}}$ and $\det \boldsymbol{Q} = 1$. Here, only rigid body motions excluding reflections are considered and hence $\det \boldsymbol{Q} = 1$. The transformation in Equation (1.179) conserves the distance and angles. Indeed,

$$\mathrm{d}\boldsymbol{x}' = \boldsymbol{Q} \cdot \mathrm{d}\boldsymbol{x} \tag{1.181}$$

and consequently

$$(\mathrm{d}s')^2 = \mathrm{d}\boldsymbol{x}' \cdot \mathrm{d}\boldsymbol{x}' = \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{Q} \cdot \mathrm{d}\boldsymbol{x} = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x} = \mathrm{d}s^2 \tag{1.182}$$

and

$$\mathrm{d}\boldsymbol{x}' \cdot \mathrm{d}\boldsymbol{y}' = \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{Q} \cdot \mathrm{d}\boldsymbol{y} = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{y}. \tag{1.183}$$

It is generally accepted that material properties should be independent of the coordinate frame of the observer. Hence, in describing these material properties, we would like to use quantities that ensure that the frame independence is guaranteed. For a time-independent rigid body motion, it is known that vectors $\boldsymbol{a}$ and second-order tensors $\boldsymbol{b}$ in the spatial description transform according to

$$\boldsymbol{a}' = \boldsymbol{Q} \cdot \boldsymbol{a} \tag{1.184}$$

and

$$\boldsymbol{b}' = \boldsymbol{Q} \cdot \boldsymbol{b} \cdot \boldsymbol{Q}^{\mathrm{T}}. \tag{1.185}$$

Requiring this to be true for time-dependent rigid motions guarantees the spatial frame indifference of any material law using such quantities. Vectors and tensors obeying Equation (1.184) and Equation (1.185) for time-dependent rigid body motions are called *objective*. From Equation (1.181), it is clear that $\mathrm{d}\boldsymbol{x}$ is objective while time-differentiation of Equation (1.179) reveals that the velocity $\boldsymbol{v}$ and the acceleration are not:

$$\boldsymbol{v}' = \dot{\boldsymbol{Q}} \cdot \boldsymbol{x} + \boldsymbol{Q} \cdot \boldsymbol{v} \tag{1.186}$$

$$\boldsymbol{a}' = \ddot{\boldsymbol{Q}} \cdot \boldsymbol{x} + 2\dot{\boldsymbol{Q}} \cdot \boldsymbol{v} + \boldsymbol{Q} \cdot \boldsymbol{a}. \tag{1.187}$$

Accordingly, $\boldsymbol{v}$ and $\boldsymbol{a}$ should not be used to describe material laws. That the acceleration is not objective is well known and is the reason for the Coriolis force in mechanics. Since the transformation in Equation (1.179) conserves the distance, one obtains (Equation (1.164)):

$$\begin{aligned}\frac{D}{Dt}(\mathrm{d}s')^2 &= 2\mathrm{d}\boldsymbol{x}' \cdot \boldsymbol{d}' \cdot \mathrm{d}\boldsymbol{x}' \\ &= 2\mathrm{d}\boldsymbol{x} \cdot \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{d}' \cdot \boldsymbol{Q} \cdot \mathrm{d}\boldsymbol{x} \\ &= \frac{D}{Dt}\mathrm{d}s^2 = 2\mathrm{d}\boldsymbol{x} \cdot \boldsymbol{d} \cdot \mathrm{d}\boldsymbol{x}\end{aligned} \tag{1.188}$$

and consequently,

$$\boldsymbol{d} = \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{d}' \cdot \boldsymbol{Q}. \tag{1.189}$$

This shows that the deformation rate tensor is objective. Notice that a second-order tensor $\boldsymbol{a}$, which maps an objective vector $\boldsymbol{b}$ into another objective vector $\boldsymbol{c}$, is objective. Indeed,

$$\boldsymbol{c}' = \boldsymbol{a}' \cdot \boldsymbol{b}' \tag{1.190}$$

implies

$$\boldsymbol{c} = (\boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{a}' \cdot \boldsymbol{Q}) \cdot \boldsymbol{b} \tag{1.191}$$

yielding

$$\boldsymbol{a} = \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{a}' \cdot \boldsymbol{Q}. \tag{1.192}$$

The time derivative of an objective vector or tensor is generally not objective. Indeed, time differentiation of Equation (1.184) and Equation (1.185) yields

$$\dot{\boldsymbol{a}}' = \underline{\dot{\boldsymbol{Q}} \cdot \boldsymbol{a}} + \boldsymbol{Q} \cdot \dot{\boldsymbol{a}} \tag{1.193}$$

$$\dot{\boldsymbol{b}}' = \underline{\dot{\boldsymbol{Q}} \cdot \boldsymbol{b} \cdot \boldsymbol{Q}^{\mathrm{T}}} + \boldsymbol{Q} \cdot \dot{\boldsymbol{b}} \cdot \boldsymbol{Q}^{\mathrm{T}} + \underline{\boldsymbol{Q} \cdot \boldsymbol{b} \cdot \dot{\boldsymbol{Q}}^{\mathrm{T}}}. \tag{1.194}$$

The terms that are underlined are the reason for the lack of objectivity.

Finally, all vectors and tensors in the material description (such as $\boldsymbol{C}$) are objective since they are not influenced by a change of the spatial frame of reference.

1.7 Balance Laws

Balance laws are important statements describing the conservation of some physical quantities. These quantities and the conservation thereof will be defined in the present section.

1.7.1 Conservation of mass

Each object in space is assigned a strictly positive scalar quantity called the *mass*. The mass is assumed to be continuously distributed, which allows for the definition of density $\rho_0(\boldsymbol{X}, t)$ by letting the volume containing particle $\boldsymbol{X}$ go to zero:

$$\rho_0(\boldsymbol{X}) := \lim_{\Delta V_0 \to 0} \frac{\Delta M}{\Delta V_0}, \quad \boldsymbol{X} \in \Delta V_0. \tag{1.195}$$

ΔV_0 is the volume the particle occupies in the reference configuration at time $t = t_0$. The density can change during the motion of a body. The density of a particle at time t originally at $\boldsymbol{X}$ is

$$\rho(\boldsymbol{X}, t) := \lim_{\Delta V \to 0} \frac{\Delta M}{\Delta V}, \quad \boldsymbol{x}(\boldsymbol{X}, t) \in \Delta V. \tag{1.196}$$

ΔV is the volume the particle occupies at time t in the spatial configuration. The axiom of the conservation of mass now states that “the time rate of change of the total mass of a body is zero”. Accordingly,

$$\frac{D}{Dt}\left(\int_V \rho \,\mathrm{d}v\right) = 0. \tag{1.197}$$

1.7.2 Conservation of momentum

The momentum (also called *linear momentum*) of an infinitesimal mass dm moving with a velocity $\boldsymbol{v}$ is defined as

$$\boldsymbol{v}\,\mathrm{d}m = \rho\boldsymbol{v}\,\mathrm{d}v. \tag{1.198}$$

The principle of conservation of momentum states that “the time rate of change of linear momentum is equal to the total force $\boldsymbol{F}$ acting on a body”. Forces acting on a body are either body forces $\boldsymbol{F}_\mathrm{b}$ resulting from distant actions such as gravity, surface tractions $\boldsymbol{F}_\mathrm{s}$ resulting from immediate contact such as classical friction forces, or concentrated forces $\boldsymbol{F}_\mathrm{c}$. Enough continuity is assumed such that the body force per unit volume $\boldsymbol{f}$ and the force per unit area $\boldsymbol{t}_{(\boldsymbol{n})}$ can be defined as follows:

$$\mathrm{d}\boldsymbol{F}_\mathrm{b} =: \rho\boldsymbol{f}\,\mathrm{d}v \tag{1.199}$$

$$\mathrm{d}\boldsymbol{F}_\mathrm{s} =: \boldsymbol{t}_{(\boldsymbol{n})}\,\mathrm{d}a. \tag{1.200}$$

Accordingly,

$$\frac{D}{Dt}\int_V \rho\boldsymbol{v}\,\mathrm{d}v = \oint_A \boldsymbol{t}_{(\boldsymbol{n})}\,\mathrm{d}a + \int_V \rho\boldsymbol{f}\,\mathrm{d}v + \sum \boldsymbol{F}_\mathrm{c} \tag{1.201}$$

where A denotes the surface of the body at stake. This principle is also known as *Newton’s second law*.

1.7.3 Conservation of angular momentum

The angular momentum of a particle with mass dm, velocity $\boldsymbol{v}$ and location $\boldsymbol{x}$ is defined as

$$\boldsymbol{x} \times \boldsymbol{v}\,\mathrm{d}m \tag{1.202}$$

where × symbolizes the vector product (also called the *cross product*) of two vectors. The vector product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is a one-form $\boldsymbol{c}$ satisfying

$$\boldsymbol{c} \cdot \boldsymbol{a} = \boldsymbol{c} \cdot \boldsymbol{b} = 0 \tag{1.203}$$

and

$$\boldsymbol{c} \cdot \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{a})(\boldsymbol{b} \cdot \boldsymbol{b}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2. \tag{1.204}$$

Accordingly, $\boldsymbol{g}_i \times \boldsymbol{g}_j$ is proportional to $\boldsymbol{g}^k$. The proportionality constant λ can be determined from Equation (1.204):

$$\lambda^2 g^{kk} = g_{ii} g_{jj} - (g_{ij})^2 = \text{cofactor}(g_{kk}). \tag{1.205}$$

Since $\boldsymbol{g}^\sharp$ is the inverse of $\boldsymbol{g}^\flat$, one finds

$$g^{kk} = \frac{\text{cofactor}(g_{kk})}{\det \boldsymbol{g}^\flat} \tag{1.206}$$

leading to

$$\boldsymbol{g}_i \times \boldsymbol{g}_j = e_{ijk} \boldsymbol{g}^k \sqrt{\det \boldsymbol{g}^\flat}. \tag{1.207}$$

Similarly, the moment of a force $\boldsymbol{F}$ acting at a location $\boldsymbol{x}$ is defined as $\boldsymbol{x} \times \boldsymbol{F}$. The principle of conservation of angular momentum states that "the time rate of change of angular momentum is equal to the total moment due to forces and couples acting on the body". Hence,

$$\frac{D}{Dt} \int_V \rho \boldsymbol{x} \times \boldsymbol{v}\,\mathrm{d}v = \oint_A \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{n})}\,\mathrm{d}a + \int_V \rho \boldsymbol{x} \times \boldsymbol{f}\,\mathrm{d}v + \sum \boldsymbol{x} \times \boldsymbol{F}_{\mathbf{c}} + \sum \boldsymbol{M}_{\mathbf{c}}. \tag{1.208}$$

Here $\boldsymbol{M}_{\mathbf{c}}$ represents concentrated moments. It is assumed that there are no distributed moments, which essentially means that this treatise is limited to nonpolar theories. Readers interested in polar theories (used, for example, for the description of liquid crystals) are referred to (Eringen 1980).

1.7.4 Conservation of energy

This principle states that "the time rate of change of the sum of the kinetic energy $\mathcal{K}$ and internal energy $\mathcal{E}$ is equal to the sum of the work rate of all forces and couples $\mathcal{W}$ acting on the body and all other energies $\mathcal{U}$ that enter or leave the body per unit time". The total kinetic energy of a body is defined by

$$\mathcal{K} = \frac{1}{2} \int_V \rho \boldsymbol{v} \cdot \boldsymbol{v}\,\mathrm{d}v \tag{1.209}$$

and the rate of work of all forces and couples by

$$\mathcal{W} = \oint_A \boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v}\,\mathrm{d}a + \int_V \rho \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}v + \sum \boldsymbol{F}_\mathbf{c} \cdot \boldsymbol{v}_\mathbf{c} + \sum \boldsymbol{M}_\mathbf{c} \cdot \boldsymbol{\omega}_\mathbf{c} \tag{1.210}$$

where $\boldsymbol{\omega}_\mathbf{c}$ is the angular velocity of the particle $\boldsymbol{M}_c$ is acting.

The internal energy is a new quantity. It is assumed that it is continuously distributed such that the energy density or energy per unit mass ε can be defined as

$$\mathcal{E} = \int_V \rho\varepsilon\,\mathrm{d}v. \tag{1.211}$$

Other energies can, for example, be of thermal, chemical or electromagnetic origin. Here we limit the discussion to thermal energy. In that case, $\mathcal{U}$ amounts to

$$\mathcal{U} = -\oint_A \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a} + \int_V \rho h\,\mathrm{d}v + \sum H_\mathrm{c} \tag{1.212}$$

where $\boldsymbol{q}$ is the heat flux through area $\mathrm{d}\boldsymbol{a}$ (the minus sign implies that the body is losing energy if $\boldsymbol{q}$ points outwards), h is the body heat density and H_c is the heat due to concentrated heat sources. Consequently, the principle of conservation of energy reads

$$\frac{D}{Dt}\int_V \left(\rho\varepsilon + \frac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}\right)\mathrm{d}v = \oint_A (\boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v} - \boldsymbol{q}\cdot\boldsymbol{n})\,\mathrm{d}a$$

$$+ \int_V (\rho\boldsymbol{f}\cdot\boldsymbol{v} + \rho h)\,\mathrm{d}v + \sum H_\mathrm{c} + \sum \boldsymbol{F}_\mathbf{c} \cdot \boldsymbol{v}_\mathbf{c} + \sum \boldsymbol{M}_\mathbf{c} \cdot \boldsymbol{\omega}_\mathbf{c}. \tag{1.213}$$

This is also called the *first law of thermodynamics.*

1.7.5 Entropy inequality

This principle, also called the *second law of thermodynamics* or Clausius–Duhem inequality, states that “the time rate of change of the entropy H of a body is never less than the sum of the entropy $\boldsymbol{s}$ entering the body through its surface and the entropy B generated by body sources”. Hence,

$$\frac{DH}{Dt} \geq B + \oint_A \boldsymbol{s}\cdot\mathrm{d}\boldsymbol{a}. \tag{1.214}$$

Defining the entropy density η and the entropy source density b by

$$H = \int_V \rho\eta\,\mathrm{d}v \tag{1.215}$$

and

$$B = \int_V \rho b\,\mathrm{d}v \tag{1.216}$$

one finds

$$\frac{D}{Dt}\int_V \rho\eta\,\mathrm{d}v \geq \int_V \rho b\,\mathrm{d}v + \oint_A \boldsymbol{s}\cdot\mathrm{d}\boldsymbol{a}. \tag{1.217}$$

Notice that this is an inequality. If other phenomena are considered such as electromagnetic actions, additional laws apply. Here we concentrate on thermomechanical processes.

1.7.6 Closure

At first sight, the formulation of the balance laws does not look very promising for our primary goal, that is, the determination of $\boldsymbol{x}(\boldsymbol{X}, t)$. Indeed, a lot of extra unknowns have been defined: ρ, $\boldsymbol{t}_{(\boldsymbol{n})}$, ε, $\eta, \ldots$ On the other hand, some of the new variables are formulated in terms of previously defined unknowns such as $\mathcal{K}(\boldsymbol{v})$. The relevance of the balance laws is based on the relationship they establish with the physical world through quantities such as f and h. They are fundamental axioms based on physical observations and as such indispensable. The extra unknowns will be taken care of later on by the material description (constitutive equations).

1.8 Localization of the Balance Laws

An important notion in the classical theory of continuum mechanics is the localization of the balance laws. In the previous section, the balance laws were formulated for finite bodies. The localization principle postulates that the balance laws are valid for any body, no matter how small. This strong assumption leads to a differential form of the balance laws. Nonlocal theories exist (Eringen 1976), which do not make this assumption but rather assume a sphere of influence for every point.

1.8.1 Conservation of mass

Since $\mathrm{d}v = J\,\mathrm{d}V$, one can write Equation (1.197) as

$$\frac{D}{Dt}\left(\int_{V_0} \rho J \,\mathrm{d}V\right) = 0. \tag{1.218}$$

V_0 is the volume of the mass at time $t = t_0$ and as such not dependent on time. Hence, Equation (1.218) is equivalent to

$$\int_{V_0} \frac{D}{Dt}(\rho J)\,\mathrm{d}V = 0. \tag{1.219}$$

Since this equation must be satisfied for any volume, the balance of mass yields

$$\frac{D}{Dt}(\rho J) = 0 \tag{1.220}$$

which can also be written as (Equation (1.178))

$$\rho \nabla \cdot \boldsymbol{v} + \frac{D\rho}{Dt} = 0 \tag{1.221}$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \boldsymbol{v} + \rho \nabla \cdot \boldsymbol{v} = 0 \tag{1.222}$$

which is equivalent to

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{1.223}$$

1.8.2 Conservation of momentum

Equation (1.201) can be written as

$$\int_{V_0} \frac{D}{Dt}(\rho J \boldsymbol{v})\,\mathrm{d}V = \oint_A \boldsymbol{t}_{(\boldsymbol{n})}\,\mathrm{d}a + \int_{V_0} \rho J \boldsymbol{f}\,\mathrm{d}V + \sum \boldsymbol{F}_{\mathbf{c}}. \tag{1.224}$$

Before localization can be applied to Equation (1.224) the surface integral in the right-hand side has to be converted to a volume integral. To this end, the original conservation of momentum Equation (1.201) is applied to the volume in Figure 1.5. In addition, the mean value theorem is used, stating that for a continuous function ϕ in a domain Ω a point $\boldsymbol{x}^* \in \Omega$ exists such that

$$\int_{\Omega} \phi(\boldsymbol{x})\,\mathrm{d}\Omega = \phi(\boldsymbol{x}^*) \int_{\Omega} \mathrm{d}\Omega. \tag{1.225}$$

Hence

$$\frac{D}{Dt}(\rho^* \boldsymbol{v}^* \Delta v) = \boldsymbol{t}_{(\boldsymbol{n})}\Delta a + \boldsymbol{t}_{(-\boldsymbol{n}^k)}\Delta a_k + \rho^* \boldsymbol{f}^* \Delta v \tag{1.226}$$

assuming there are no point loads in the volume Δv. Newton's third law (action = reaction) dictates that

$$\boldsymbol{t}_{(-\boldsymbol{n}^k)} = -\boldsymbol{t}_{(\boldsymbol{n}^k)} \tag{1.227}$$

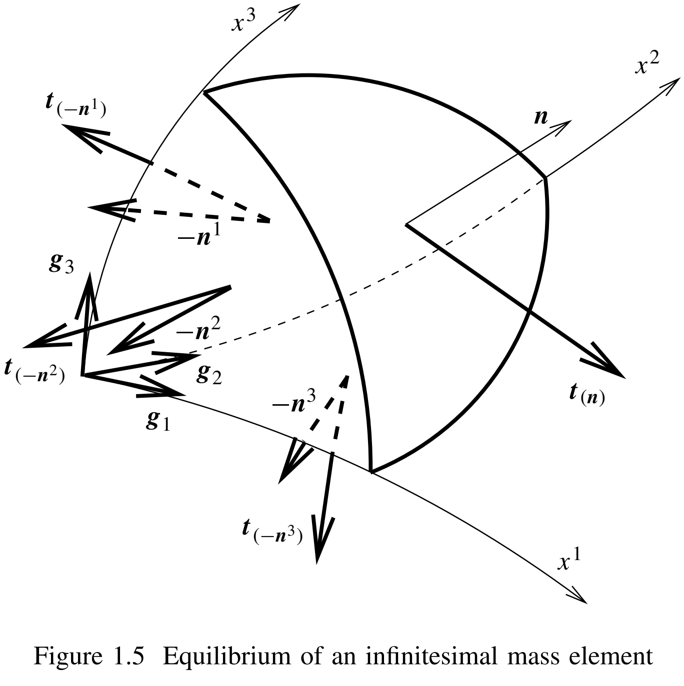

Figure 1.5 Equilibrium of an infinitesimal mass element

and Equation (1.226) reduces to

$$\frac{D}{Dt}(\rho^* \boldsymbol{v}^* \Delta v) = \boldsymbol{t}_{(\boldsymbol{n})} \Delta a - \boldsymbol{t}_{(\boldsymbol{n}^k)} \Delta a_k + \rho^* \boldsymbol{f}^* \Delta v. \tag{1.228}$$

Notice that $\boldsymbol{n}^1$, $\boldsymbol{n}^2$ and $\boldsymbol{n}^3$ are positive in the direction of $\boldsymbol{g}^1$, $\boldsymbol{g}^2$ and $\boldsymbol{g}^3$ respectively. Since in the limit $\Delta v \to 0$

$$\lim_{\Delta v \to 0} \frac{\Delta v}{\Delta a} = 0 \tag{1.229}$$

for an infinitesimal volume, Equation (1.226) reduces to

$$\boldsymbol{t}_{(\boldsymbol{n})} \Delta a = \boldsymbol{t}_{(\boldsymbol{n}^k)} \Delta a_k. \tag{1.230}$$

Since

$$\Delta a_k = n_k \Delta a \tag{1.231}$$

where n_k are the components of the one-form $\boldsymbol{n}$, that is, $\boldsymbol{n} = n_k \boldsymbol{g}^k$, one finds

$$\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{t}_{(\boldsymbol{n}^k)} n_k. \tag{1.232}$$

Notice that the normal to a surface is a one-form since the inner product with a length vector produces a scalar volume. Denoting the traction vector on a surface with unit normal $\boldsymbol{n}^k$ by $\boldsymbol{t}^k$, Equation (1.232) reads

$$\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{t}^k n_k. \tag{1.233}$$

Accordingly, the stress on a surface with normal $\boldsymbol{n}$ is a linear combination of the stresses on surfaces perpendicular to the coordinate axes. Substituting Equation (1.233) into Equation (1.224) and applying Cauchy's theorem , which reads

$$\oint_A \boldsymbol{t}^k n_k \,\mathrm{d}a = \int_V \boldsymbol{t}^k{}_{;k} \,\mathrm{d}v \tag{1.234}$$

one finds after localization

$$\frac{D}{Dt}(\rho J \boldsymbol{v}) = \boldsymbol{t}^k{}_{;k} J + \rho \boldsymbol{f} J \tag{1.235}$$

at locations without concentrated forces. Applying the balance of mass yields

$$\rho \frac{D\boldsymbol{v}}{Dt} = \boldsymbol{t}^k{}_{;k} + \rho \boldsymbol{f} \tag{1.236}$$

or

$$\rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \otimes \nabla) \cdot \boldsymbol{v} \right] = \boldsymbol{t}^k{}_{;k} + \rho \boldsymbol{f}. \tag{1.237}$$

1.8.3 Conservation of angular momentum

Localization of Equation (1.208) at points without concentrated forces nor moments, taken Equation (1.232) into account, yields

$$\frac{D}{Dt}(\rho J\boldsymbol{x}\times\boldsymbol{v}) = J(\boldsymbol{x}\times\boldsymbol{t}^k)_{,k} + \rho J\boldsymbol{x}\times\boldsymbol{f} \tag{1.238}$$

or

$$\frac{D}{Dt}(\rho J)\boldsymbol{x}\times\boldsymbol{v} + J\boldsymbol{x}\times\rho\frac{D\boldsymbol{v}}{Dt} = J\boldsymbol{x}_{,k}\times\boldsymbol{t}^k + J\boldsymbol{x}\times\boldsymbol{t}^k_{,k} + J\boldsymbol{x}\times\rho\boldsymbol{f}. \tag{1.239}$$

Using the balance of mass, Equation (1.220), and the balance of momentum, Equation (1.236), yields

$$\boldsymbol{g}_k\times\boldsymbol{t}^k = 0 \tag{1.240}$$

since $\boldsymbol{x}_{,k} = \boldsymbol{g}_k$. The meaning of Equation (1.240) will become clear in Section 1.9.

1.8.4 Conservation of energy

Similar operations as in the previous section convert Equation (1.213) into

$$\frac{D}{Dt}\left(\rho J\varepsilon + \frac{1}{2}\rho J\boldsymbol{v}\cdot\boldsymbol{v}\right) = J(\boldsymbol{v}\cdot\boldsymbol{t}^k)_{;k} - \nabla\cdot\boldsymbol{q}J + \rho J\boldsymbol{f}\cdot\boldsymbol{v} + \rho Jh \tag{1.241}$$

or

$$\rho J\frac{D\varepsilon}{Dt} + J\boldsymbol{v}\cdot\rho\frac{D\boldsymbol{v}}{Dt} = J\boldsymbol{v}\cdot\boldsymbol{t}^k_{;k} + J\boldsymbol{v}_{;k}\cdot\boldsymbol{t}^k - J\nabla\cdot\boldsymbol{q} + J\boldsymbol{v}\cdot\rho\boldsymbol{f} + J\rho h. \tag{1.242}$$

Application of the balance of momentum finally leads to

$$\rho\frac{D\varepsilon}{Dt} = \boldsymbol{v}_{;k}\cdot\boldsymbol{t}^k - \nabla\cdot\boldsymbol{q} + \rho h. \tag{1.243}$$

Equation (1.243) shows that the change of the internal energy per unit of time is balanced by the stress power ($\boldsymbol{v}_{;k}\cdot\boldsymbol{t}^k$), the heat influx ($-\nabla\cdot\boldsymbol{q}$) and the heat source power (ρh).

1.8.5 Entropy inequality

Along the same lines Equation (1.217) is reduced to

$$\rho\frac{D\eta}{Dt} \geq \rho b + \nabla\cdot\boldsymbol{s}. \tag{1.244}$$

1.9 The Stress Tensor

In the previous section, it was explained that $\boldsymbol{t}^k(\boldsymbol{x})$ is the stress on an infinitesimal surface at $\boldsymbol{x}$ perpendicular to $\boldsymbol{g}_k$. The components σ^{kl} of $\boldsymbol{t}^k$ are defined by

$$\boldsymbol{t}^k = \sigma^{kl}\boldsymbol{g}_l. \tag{1.245}$$

Now, $\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{t}^k n_k$ can be rewritten as

$$\boldsymbol{t}_{(\boldsymbol{n})} = \sigma^{kl} n_k \boldsymbol{g}_l \tag{1.246}$$

or, since $n_k = \boldsymbol{g}_k \cdot \boldsymbol{n}$,

$$\boldsymbol{t}_{(\boldsymbol{n})} = \sigma^{kl} \boldsymbol{g}_l \cdot (\boldsymbol{g}_k \cdot \boldsymbol{n}) = (\sigma^{kl} \boldsymbol{g}_l \otimes \boldsymbol{g}_k) \cdot \boldsymbol{n} \tag{1.247}$$

which shows that σ^{kl} is a second-order contravariant tensor (the so-called Cauchy stress tensor) and that the stress vector on an infinitesimal surface perpendicular to $\boldsymbol{n}$ can be obtained by the scalar product of the transpose of the stress tensor at that point with $\boldsymbol{n}$, in component notation:

$$t^l_{(\boldsymbol{n})} = \sigma^{kl} n_k. \tag{1.248}$$

The Cauchy stress is also called the *true stress* since it is defined in the spatial state of reference. It is the stress the deformed state truly experiences.

An important property of σ^{kl} follows from Equation (1.240) in component notation:

$$e_{ijl} {g_k}^j \sigma^{kl} = 0 \tag{1.249}$$

where e_{ijl} is the alternating symbol. Since ${g_k}^j = {\delta_k}^j$ one finds

$$\sigma^{kl} = \sigma^{lk} \tag{1.250}$$

that is, the stress tensor is symmetric. Letting

$$\boldsymbol{\sigma} := \sigma^{kl} \boldsymbol{g}_k \otimes \boldsymbol{g}_l \tag{1.251}$$

Equation (1.250) is equivalent to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \tag{1.252}$$

and $\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \boldsymbol{n}$, Equation (1.247), is transformed into $\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{\sigma} \cdot \boldsymbol{n}$.

For the special case of $\boldsymbol{t}^k$, Equation (1.247) reduces to

$$\boldsymbol{t}^k = \boldsymbol{\sigma}^{\mathrm{T}} \cdot \boldsymbol{g}^k. \tag{1.253}$$

The term $\boldsymbol{v}_{,k} \cdot \boldsymbol{t}^k$ in the energy balance, Equation (1.243), becomes (see also Equation (1.163))

$$\begin{aligned} \boldsymbol{v}_{,k} \cdot \boldsymbol{t}^k &= \boldsymbol{g}_k \cdot (\nabla \otimes \boldsymbol{v}) \cdot \boldsymbol{\sigma}^{\mathrm{T}} \cdot \boldsymbol{g}^k \\ &= (\boldsymbol{v} \otimes \nabla) : \boldsymbol{\sigma}^{\mathrm{T}} = (\boldsymbol{v} \otimes \nabla) : \boldsymbol{\sigma} \end{aligned} \tag{1.254}$$

yielding for the complete energy equation

$$\rho \frac{D\varepsilon}{Dt} = (\boldsymbol{v} \otimes \nabla) : \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{q} + \rho h. \tag{1.255}$$

Using the definition in Equation (1.166)

$$\rho \frac{D\varepsilon}{Dt} = \boldsymbol{l} : \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{q} + \rho h \tag{1.256}$$

or

$$\rho \frac{D\varepsilon}{Dt} = \boldsymbol{d} : \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{q} + \rho h. \tag{1.257}$$

Since $\boldsymbol{\sigma}$ is symmetric, all its eigenvalues are real. The meaning of the eigenvalues can be clarified by looking for the maximum normal stress in a point. Since $\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot \boldsymbol{\sigma}$, the normal stress σ on an infinitesimal surface with normal $\boldsymbol{n}$ is given by

$$\sigma = \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}. \tag{1.258}$$

Maximizing σ with the constraint $\|\boldsymbol{n}\| = \boldsymbol{n} \cdot \boldsymbol{g}^{\sharp} \cdot \boldsymbol{n} = 1$ yields

$$\frac{\partial}{\partial \boldsymbol{n}} \left[\boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} - \lambda (\boldsymbol{n} \cdot \boldsymbol{g}^{\sharp} \cdot \boldsymbol{n}) \right] = 0. \tag{1.259}$$

$\boldsymbol{g}^{\sharp}$ is the contravariant metric tensor whose components g^{kl} satisfy

$$g^{kl} = \boldsymbol{g}^{k} \cdot \boldsymbol{g}^{l}. \tag{1.260}$$

Equation (1.259) leads to the eigenvalue problem

$$(\boldsymbol{\sigma} - \lambda \boldsymbol{g}^{\sharp}) \cdot \boldsymbol{n} = 0. \tag{1.261}$$

Similar to Equation (1.121), one can write

$$\boldsymbol{\sigma} = \sum_{i=1}^{3} \lambda_{i\sigma} (\boldsymbol{n}_i \otimes \boldsymbol{n}_i) \tag{1.262}$$

where $\boldsymbol{n}_i$ are the complementary basis vectors to the eigen one-forms of $\boldsymbol{\sigma}$. However, contrary to $\boldsymbol{C}$ the tensor $\boldsymbol{\sigma}$ is not positive definite, since σ in Equation (1.258) can be negative (pressure). In general, the stress eigenvectors do not coincide with the strain eigenvectors. Consequently, $\boldsymbol{n}_i$ in Equation (1.262) is usually distinct from $\boldsymbol{n}_i$ in Equation (1.136).

The force on an infinitesimal area $\mathrm{d}a$ can be written as

$$\begin{aligned} \mathrm{d}\boldsymbol{F} = \boldsymbol{t}_{(\boldsymbol{n})}\, \mathrm{d}a &= \boldsymbol{\sigma} \cdot \boldsymbol{n}\, \mathrm{d}a \\ &= \boldsymbol{\sigma} \cdot \mathrm{d}\boldsymbol{a} \\ &= \boldsymbol{\sigma} \cdot J \boldsymbol{F}^{-\mathrm{T}} \cdot \mathrm{d}\boldsymbol{A} \\ &= J\boldsymbol{\sigma} \cdot (\boldsymbol{F}^{-\mathrm{T}} \cdot \boldsymbol{N})\, \mathrm{d}A \\ &=: \boldsymbol{T}_{(\boldsymbol{N})}\, \mathrm{d}A \end{aligned} \tag{1.263}$$

where Equation (1.65) was used. The vector $\boldsymbol{T}_{(\boldsymbol{N})}$ represents an equivalent stress vector on the surface in the reference configuration and satisfies

$$\boldsymbol{T}_{(\boldsymbol{N})} = J\boldsymbol{\sigma} \cdot \boldsymbol{F}^{-\mathrm{T}} \cdot \boldsymbol{N}. \tag{1.264}$$

Defining the Piola–Kirchhoff tensor of the first kind by an expression similar to Equation (1.247):

$$\boldsymbol{T}_{(\boldsymbol{N})} := \boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{N} \tag{1.265}$$

one finds

$$\boldsymbol{P} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}. \tag{1.266}$$

Notice that $\boldsymbol{P}$ is a two-point tensor, in component notation:

$$P^{Kk} = JX^{K}{}_{,l}\sigma^{lk}. \tag{1.267}$$

The tensor $\boldsymbol{P}$ is not symmetric. Indeed, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ is equivalent to

$$\boldsymbol{F} \cdot \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{F}^{\mathrm{T}}. \tag{1.268}$$

To remediate this, a Piola–Kirchhoff stress tensor of the second kind, $\boldsymbol{S}$, is defined by

$$\boldsymbol{S} := \boldsymbol{P} \cdot \boldsymbol{F}^{-\mathrm{T}} = J\boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-\mathrm{T}}. \tag{1.269}$$

This tensor is symmetric and satisfies

$$\boldsymbol{S} = S^{KL}\boldsymbol{G}_K \otimes \boldsymbol{G}_L. \tag{1.270}$$

One also defines the Kirchhoff stress τ by

$$\boldsymbol{\tau} := J\boldsymbol{\sigma}. \tag{1.271}$$

Equation (1.257) can now also be written as

$$\rho_0 \frac{D\varepsilon}{Dt} = \boldsymbol{d} : \boldsymbol{\tau} - J\nabla \cdot \boldsymbol{q} + \rho_0 h. \tag{1.272}$$

In the balance equations in the previous section, a couple of quantities were defined on surfaces in the spatial configuration such as the heat vector $\boldsymbol{q}$. Similar to the derivation in Equation (1.263), an equivalent quantity in the reference configuration is defined by

$$\boldsymbol{q} \cdot \mathrm{d}\boldsymbol{a} = \boldsymbol{Q} \cdot \mathrm{d}\boldsymbol{A} \tag{1.273}$$

yielding

$$\boldsymbol{Q} = J\boldsymbol{q} \cdot \boldsymbol{F}^{-\mathrm{T}}. \tag{1.274}$$

Analogously, one defines

$$\boldsymbol{S} = J\boldsymbol{s} \cdot \boldsymbol{F}^{-\mathrm{T}} \tag{1.275}$$

for the entropy flux. Do not confuse the infinitesimal length $\mathrm{d}S$, Equation (1.8), with the Piola–Kirchhoff stress tensor of the second kind $\boldsymbol{S}$, Equation (1.269), and the entropy vector $\boldsymbol{S}$, Equation (1.275). The context should clarify what is meant.

1.10 The Balance Laws in Material Coordinates

In Sections 1.7 and 1.8, the balance laws were derived in spatial coordinates. In some cases (think of objective quantities), it is advantageous to work in material coordinates. The material form can be derived by starting over with the global form, or by simply converting spatial quantities into material quantities.

1.10.1 Conservation of mass

The spatial form $D(\rho J)/Dt$ can be trivially converted by integration into

$$\rho J = \rho_0. \tag{1.276}$$

Substitution into the spatial form yields

$$\frac{D}{Dt}(\rho_0) = 0. \tag{1.277}$$

1.10.2 Conservation of momentum

Substitution of Equation (1.253) into Equation (1.236) yields the spatial form

$$(\boldsymbol{\sigma}^{\mathrm{T}} \cdot \boldsymbol{g}^k)_{;k} + \rho \boldsymbol{f} = \rho \frac{D\boldsymbol{v}}{Dt} \tag{1.278}$$

or

$$\left[\sigma^{ml}(\boldsymbol{g}_l \otimes \boldsymbol{g}_m) \cdot \boldsymbol{g}^k)\right]_{;k} + \rho \boldsymbol{f} = \rho \frac{D\boldsymbol{v}}{Dt} \tag{1.279}$$

yielding

$$(\sigma^{kl}\boldsymbol{g}_l)_{;k} + \rho \boldsymbol{f} = \rho \frac{D\boldsymbol{v}}{Dt} \tag{1.280}$$

in component form

$$\sigma^{kl}{}_{;k} + \rho f^l = \rho \frac{Dv^l}{Dt}. \tag{1.281}$$

Notice that the semicolon in Equation (1.281) stands for the covariant differentiation of a second-order tensor ($\boldsymbol{g}_m$ in Equation (1.280) is not constant in space). For second-order contravariant tensors, one finds

$$\begin{aligned}
\boldsymbol{\sigma}_{,m} &= (\sigma^{kl}\boldsymbol{g}_k \otimes \boldsymbol{g}_l)_{,m} \\
&= \sigma^{kl}{}_{,m}\boldsymbol{g}_k \otimes \boldsymbol{g}_l + \sigma^{kl}\frac{\partial \boldsymbol{g}_k}{\partial x^m} \otimes \boldsymbol{g}_l + \sigma^{kl}\boldsymbol{g}_k \otimes \frac{\partial \boldsymbol{g}_l}{\partial x^m} \\
&= \sigma^{kl}{}_{,m}\boldsymbol{g}_k \otimes \boldsymbol{g}_l + \sigma^{kl}\frac{\partial^2 z^p}{\partial x^m \partial x^k}\frac{\partial x^q}{\partial z^p}\boldsymbol{g}_q \otimes \boldsymbol{g}_l + \sigma^{kl}\boldsymbol{g}_k \otimes \frac{\partial^2 z^p}{\partial x^m \partial x^l}\frac{\partial x^q}{\partial z^p}\boldsymbol{g}_q \\
&= \left[\sigma^{kl}{}_{,m} + \sigma^{ql}\frac{\partial^2 z^p}{\partial x^m \partial x^q}\frac{\partial x^k}{\partial z^p} + \sigma^{kq}\frac{\partial^2 z^p}{\partial x^m \partial x^q}\frac{\partial x^l}{\partial z^p}\right]\boldsymbol{g}_k \otimes \boldsymbol{g}_l.
\end{aligned} \tag{1.282}$$

Hence,

$$\sigma^{kl}{}_{;m} = \sigma^{kl}{}_{,m} + \sigma^{ql}\begin{Bmatrix} k \\ mq \end{Bmatrix} + \sigma^{kq}\begin{Bmatrix} l \\ mq \end{Bmatrix} \tag{1.283}$$

where the braces denote the Christoffel symbols of the second kind (cf Equation (1.80)).

Substituting Equation (1.267) and Equation (1.276) into Equation (1.281) yields

$$J(J^{-1}x^k_{,K}P^{Kl})_{;k} + \rho_0 f^l = \rho_0 \frac{Dv^l}{Dt} \tag{1.284}$$

or

$$x^k_{,K}(P^{Kl})_{;k} + \rho_0 f^l = \rho_0 \frac{Dv^l}{Dt} \tag{1.285}$$

since $(J^{-1}x^k_{,K})_{,k} = 0$. This identity can be obtained by realizing that J^{-1} is the Jacobian determinant of the inverse deformation $\boldsymbol{X}(\boldsymbol{x})$. Consequently, Equation (1.59) becomes

$$\begin{aligned} x^k_{,K} &= \frac{1}{J^{-1}}\text{cofactor } (X^K_{,k}) \\ &= \tfrac{1}{2} J e^{klm} e_{KLM} X^L_{,l} X^M_{,m} \end{aligned} \tag{1.286}$$

Accordingly, focusing on rectangular coordinates for simplicity,

$$(J^{-1}x^k_{,K})_{,k} = \tfrac{1}{2} e^{klm} e_{KLM}(X^L_{,lk}X^M_{,m} + X^L_{,l}X^M_{,mk}) = 0 \tag{1.287}$$

since by switching l and k or m and k the permutation symbols change sign. Another derivation is given by (Ogden 1984). Equation (1.285) finally yields

$$P^{Kl}_{;K} + \rho_0 f^l = \rho_0 \frac{Dv^l}{Dt}. \tag{1.288}$$

Notice that the first term involves covariant differentiation of a contravariant two-point tensor. One finds (Eringen 1975)

$$\begin{aligned} (P^{Kk}\boldsymbol{G}_K \otimes \boldsymbol{g}_k)_{,L} &= P^{Kk}_{,L}\boldsymbol{G}_K \otimes \boldsymbol{g}_k + P^{Kk}\frac{\partial \boldsymbol{G}_K}{\partial X^L} \otimes \boldsymbol{g}_k + P^{Kk}\boldsymbol{G}_K \otimes \frac{\partial \boldsymbol{g}_k}{\partial x^l} x^l_{,L} \\ &= \left(P^{Kk}_{,L} + P^{Kk}\frac{\partial^2 Z^M}{\partial X^L \partial X^N}\frac{\partial X^K}{\partial Z^M} + P^{Kn}\frac{\partial^2 z^m}{\partial x^l \partial x^n}\frac{\partial x^k}{\partial z^m} x^l_{,L} \right) \boldsymbol{G}_K \otimes \boldsymbol{g}_k. \end{aligned} \tag{1.289}$$

Accordingly,

$$P^{Kk}_{;L} = P^{Kk}_{,L} + P^{Nk}\begin{Bmatrix} K \\ LN \end{Bmatrix} + P^{Kn}\begin{Bmatrix} k \\ ln \end{Bmatrix} x^l_{,L}. \tag{1.290}$$

In vector form, Equation (1.288) yields

$$\nabla_0 \cdot \boldsymbol{P} + \rho_0 \boldsymbol{f} = \rho_0 \frac{D\boldsymbol{v}}{Dt} \tag{1.291}$$

or

$$\nabla_0 \cdot (\boldsymbol{S} \cdot \boldsymbol{F}^{\mathrm{T}}) + \rho_0 \boldsymbol{f} = \rho_0 \frac{D\boldsymbol{v}}{Dt} \tag{1.292}$$

where ∇_0 represents the gradient in material coordinates.

1.10.3 Conservation of angular momentum

The material form of $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ was derived in Section 1.8 and can be written as

$$\boldsymbol{F} \cdot \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} \cdot \boldsymbol{F}^{\mathrm{T}} \tag{1.293}$$

or

$$\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}} \tag{1.294}$$

depending on whether the first or second Piola–Kirchhoff stress tensor is used.

1.10.4 Conservation of energy

Since

$$\boldsymbol{q} = J^{-1} \boldsymbol{Q} \cdot \boldsymbol{F}^{\mathrm{T}} \tag{1.295}$$

$$\boldsymbol{\sigma} = J^{-1} \boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{\mathrm{T}} \tag{1.296}$$

$$\boldsymbol{d} = \boldsymbol{F}^{-\mathrm{T}} \cdot \dot{\boldsymbol{E}} \cdot \boldsymbol{F}^{-1} \tag{1.297}$$

Equation (1.257) now yields

$$\rho_0 \frac{D\varepsilon}{Dt} = \dot{\boldsymbol{E}} : \boldsymbol{S} - \nabla_0 \cdot \boldsymbol{Q} + \rho_0 h. \tag{1.298}$$

Because of the expressions for work power in Equation (1.272) and Equation (1.298), it is said that $(\boldsymbol{d}, \boldsymbol{\tau})$ and $(\dot{\boldsymbol{E}}, \boldsymbol{S})$ are conjugate pairs in the spatial and material description respectively. Indeed,

$$\boldsymbol{d} : \boldsymbol{\tau} = \dot{\boldsymbol{E}} : \boldsymbol{S}. \tag{1.299}$$

Recall that $\boldsymbol{d}$ is the push-forward tensor of $\dot{\boldsymbol{E}}$. Equivalently, $\boldsymbol{\tau}$ is called the *push-forward* of $\boldsymbol{S}$ and equivalently $\boldsymbol{S}$ is the *pullback* of $\boldsymbol{\tau}$. One obtains

$$\boldsymbol{S} = \boldsymbol{F}^{-1} \cdot \boldsymbol{\tau} \cdot \boldsymbol{F}^{-\mathrm{T}}. \tag{1.300}$$

1.10.5 Entropy inequality

Along the same lines, one obtains for the material equivalent of Equation (1.244)

$$\rho_0 \frac{D\eta}{Dt} \geq \rho_0 b + \nabla_0 \cdot \boldsymbol{S} \tag{1.301}$$

where $\boldsymbol{S}$ is the entropy flux vector in material coordinates.

The entropy inequality plays an important role in the derivation of admissible constitutive equations. For thermal processes, the entropy influx and source can be written as the corresponding heat influx and source, divided by the absolute temperature θ . Consequently,

$$\boldsymbol{S} = -\frac{\boldsymbol{Q}}{\theta} + \boldsymbol{S}_1 \tag{1.302}$$

$$b = \frac{h}{\theta} + b_1 \tag{1.303}$$

where $\boldsymbol{S}_1$ and b_1 are the entropy influx and source due to other processes respectively. For simple thermomechanical processes that are considered here, $\boldsymbol{S}_1$ and b_1 are zero. Accordingly, the entropy inequality reads

$$\rho_0 \frac{D\eta}{Dt} \geq \rho_0 \frac{h}{\theta} - \nabla_0 \cdot \frac{\boldsymbol{Q}}{\theta} \tag{1.304}$$

or

$$\rho_0 \frac{D\eta}{Dt} \geq \rho_0 \frac{h}{\theta} - \frac{1}{\theta} \nabla_0 \cdot \boldsymbol{Q} + \frac{1}{\theta^2} \boldsymbol{Q} \cdot \nabla_0 \theta. \tag{1.305}$$

Solving the energy balance Equation (1.298) for $\nabla_0 \cdot \boldsymbol{Q}$ and substituting into Equation (1.305) yields

$$\rho_0 \left(\dot{\eta} - \frac{\dot{\varepsilon}}{\theta} \right) + \frac{1}{\theta} \dot{\boldsymbol{E}} : \boldsymbol{S} - \frac{1}{\theta^2} \boldsymbol{Q} \cdot \nabla_0 \theta \geq 0. \tag{1.306}$$

This is the preferred form that will be used for the derivation of the constitutive equations.

1.11 The Weak Form of the Balance of Momentum

In this section, an alternative form of the balance of momentum will be derived, which will form the basis for much of the finite element formulation to follow in subsequent chapters. In the material formulation, the weak form is generally known as the *principle of virtual work* and in the spatial description it is known as the *virtual power principle*. It will be shown that the strong form deduced so far, Equation (1.288), is completely equivalent to the weak form by first deriving the weak form from the strong form and subsequently the strong form from the weak form. General curvilinear coordinates are assumed throughout. To obtain the equations in rectangular coordinates, replace the covariant differentiation by partial differentiation.

1.11.1 Formulation of the boundary conditions (material coordinates)

The balance of momentum in material form

$$P^{Kk}{}_{;K} + \rho_0 f^k = \rho_0 \frac{D^2 u^k}{Dt^2} \tag{1.307}$$

will be supplemented here with boundary conditions. Suppose that the material volume V_0 is surrounded by a surface A_0 consisting of internal surfaces A_{0i}, surfaces on which the displacements are described A_{0u} and surfaces on which the traction is defined, A_{0t}. Accordingly,

$$\boldsymbol{T}^+_{(\boldsymbol{N}^+)} + \boldsymbol{T}^-_{(\boldsymbol{N}^-)} = 0 \quad \text{on } A_{0i} \tag{1.308}$$

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \quad \text{on } A_{0u} \tag{1.309}$$

$$\boldsymbol{T}_{(\boldsymbol{N})} = \overline{\boldsymbol{T}}_{(\boldsymbol{N})} \quad \text{on } A_{0t}. \tag{1.310}$$

At internal surfaces the material is connected, but it might change its properties, for example, due to a change of material. Equation (1.308) is equivalent to Newton's third law: action equals reaction. The plus and minus sign denote the two sides of the internal surface. Since

$$\boldsymbol{T}_{(\boldsymbol{N})} = \boldsymbol{T}^K N_K \tag{1.311}$$

and $\boldsymbol{N}^- = -\boldsymbol{N}^+$, Equation (1.308) also reads

$$(\boldsymbol{T}^{K+} - \boldsymbol{T}^{K-})N_K^+ =: \left[\boldsymbol{T}^K\right] N_K^+ = 0. \tag{1.312}$$

This is also called the *traction continuity condition*. Note that Equation (1.307) is equivalent to

$$P^{Kk}{}_{;K}\boldsymbol{g}_k + \rho_0 f^k \boldsymbol{g}_k = \rho_0 \frac{D^2 u^k}{Dt^2}\boldsymbol{g}_k \tag{1.313}$$

and hence,

$$P^{Kk}{}_{;K} g_k{}^L + \rho_0 f^k g_k{}^L = \rho_0 \frac{D^2 u^k}{Dt^2} g_k{}^L \tag{1.314}$$

where $g_k{}^L := \boldsymbol{g}_k \cdot \boldsymbol{G}^L$ are called *shifters* since they move quantities from one coordinate system into another. Indeed, for a vector $\boldsymbol{v}$ one has

$$\boldsymbol{v} = v^k \boldsymbol{g}_k \Rightarrow \boldsymbol{v} \cdot \boldsymbol{G}^L = v^k g_k{}^L \Rightarrow V^L = v^k g_k{}^L. \tag{1.315}$$

Accordingly, Equation (1.307) is equivalent to

$$P^{Kk}{}_{;K} g_k{}^L + \rho_0 f^L = \rho_0 \frac{D^2 u^L}{Dt^2}. \tag{1.316}$$

1.11.2 Deriving the weak form from the strong form (material coordinates)

Let us consider an infinitesimal perturbation of the displacement field $\delta\boldsymbol{u}$ with components δu_k satisfying the geometric boundary conditions in Equation (1.309). Accordingly,

$$\delta\boldsymbol{u} = 0 \quad \text{on } A_{0u}. \tag{1.317}$$

Taking the scalar product of the vector Equation (1.307) with the one-form $\delta\boldsymbol{u}$ and integrating over the material volume leads to

$$\int_{V_0} \left[P^{Kk}{}_{;K} + \rho_0 \left(f^k - \frac{D^2 u^k}{Dt^2} \right) \right] \delta u_k \, \mathrm{d}V = 0. \tag{1.318}$$

Since (the usual differentiation rules also apply to covariant differentiation)

$$P^{Kk}{}_{;K}\delta u_k = (P^{Kk}\delta u_k)_{;K} - P^{Kk}\delta u_{k;K} \tag{1.319}$$

and applying Cauchy's theorem, Equation (1.234), one obtains

$$\int_{A_0} P^{Kk} N_K \delta u_k \,\mathrm{d}A - \int_{V_0} \left\{ P^{Kk} \delta u_{k;K} - \left[\rho_0 \left(f^k - \frac{D^2 u^k}{Dt^2} \right) \right] \delta u_k \right\} \mathrm{d}V = 0 \qquad (1.320)$$

or

$$\int_{V_0} P^{Kk} \delta u_{k;K} \,\mathrm{d}V = \int_{A_{0t}} \overline{T}^K_{(N)} \delta U_K \,\mathrm{d}A + \int_{V_0} \rho_0 f^K \delta U_K \,\mathrm{d}V - \int_{V_0} \rho_0 \frac{D^2 u^K}{Dt^2} \delta U_K \,\mathrm{d}V \qquad (1.321)$$

since $T^{K+}{}_{(\boldsymbol{N}^+)} + T^{K-}{}_{(\boldsymbol{N}^-)} = 0$ on A_{0i}, $\delta U_K = 0$ on A_{0u} and $\boldsymbol{T}_{(\boldsymbol{N})} = \overline{\boldsymbol{T}}_{(\boldsymbol{N})}$ on A_{0t}. Through the relationship

$$P^{Kk} = S^{KL} x^k{}_{,L} \qquad (1.322)$$

one obtains

$$P^{Kk} \delta u_{k;K} = S^{KL} x^k{}_{,L} \delta x_{k,K} = S^{KL} x^k{}_{,L} \delta x^m_{,K} g_{km} = S^{KL} \delta E_{KL}. \qquad (1.323)$$

Indeed,

$$\begin{aligned} S^{KL} \delta E_{KL} &= S^{KL} \delta \left(\tfrac{1}{2} x^k{}_{,L} x^m{}_{,K} g_{km} - \tfrac{1}{2} G_{KL} \right) \\ &= \tfrac{1}{2} S^{KL} (\delta x^k{}_{,L} x^m{}_{,K} + x^k{}_{,L} \delta x^m{}_{,K}) g_{km} \\ &= S^{KL} x^k{}_{,L} \delta x^m{}_{,K} g_{km} \end{aligned} \qquad (1.324)$$

since both S^{KL} and g_{km} are symmetric. Notice that covariant differentiation does not apply to $\boldsymbol{x}$. Indeed, from

$$\boldsymbol{u} = \boldsymbol{o} + \boldsymbol{x} - \boldsymbol{X} \qquad (1.325)$$

one obtains

$$\boldsymbol{u}_{,K} = \boldsymbol{x}_{,K} - \boldsymbol{G}_K \qquad (1.326)$$

leading to (see Equation (1.25))

$$u^k{}_{;K} = x^k{}_{,K} - g^k_K. \qquad (1.327)$$

Concluding, Equation (1.321) can also be written as

$$\int_{V_0} S^{KL} \delta E_{KL} \,\mathrm{d}V = \int_{A_{0t}} \overline{T}^K_{(N)} \delta U_K \,\mathrm{d}A + \int_{V_0} \rho_0 f^K \delta U_K \,\mathrm{d}V - \int_{V_0} \rho_0 \frac{D^2 U^K}{Dt^2} \delta U_K \,\mathrm{d}V. \qquad (1.328)$$

The left-hand side is called the *internal virtual work*, the first term on the right-hand side is the virtual work due to external tractions, the second term is due to distributed forces and the last term is due to inertia. Notice that, although all quantities in Equation (1.328) are expressed in terms of material coordinates, some are defined as a function of their spatial counterparts such as $\overline{\boldsymbol{T}}_{(\boldsymbol{N})}$ and $\boldsymbol{f}$ through $\overline{\boldsymbol{T}}_{(\boldsymbol{N})}(\boldsymbol{X}, t)\,\mathrm{d}A = \overline{\boldsymbol{t}}_{(\boldsymbol{n})}(\boldsymbol{X}, t)\,\mathrm{d}a$ and $f^K(\boldsymbol{X}, t)\boldsymbol{G}_K = f^k(\boldsymbol{X}, t)\boldsymbol{g}_k$. Hence, both $\overline{\boldsymbol{T}}_{(\boldsymbol{N})}$ and f^K are a function of the deformation. For example, if a rotating body expands because of centrifugal loads, f^K changes.

1.11.3 Deriving the strong form from the weak form (material coordinates)

Starting from the weak form in Equation (1.321) and applying Equation (1.319) and Cauchy's theorem one obtains

$$\int_{A_0} P^{Kk} N_K \delta u_k \,\mathrm{d}A - \int_{V_0} P^{Kk}{}_{;K} \delta u_k \,\mathrm{d}V - \int_{A_{0t}} \overline{T}^k_{(\boldsymbol{N})} \delta u_k \,\mathrm{d}A - \int_{V_0} \rho_0 \left(f^k - \frac{D^2 u^k}{Dt^2} \right) \delta u_k \,\mathrm{d}V = 0. \quad (1.329)$$

Since $P^{Kk} N_K = T^k_{(\boldsymbol{N})}$, $A_0 = A_{0u} \cup A_{0t} \cup A_{0i}$ and $\delta \boldsymbol{u} = 0$ for A_{0u}, one obtains

$$\int_{A_{0t}} \left(T^k_{(\boldsymbol{N})} - \overline{T}^k_{(\boldsymbol{N})} \right) \delta u_k \,\mathrm{d}A + \int_{A_{0i}} \left(T^{k+}_{(\boldsymbol{N}^+)} + T^{k-}_{(\boldsymbol{N}^-)} \right) \delta u_k \,\mathrm{d}A - \int_{V_0} \left(P^{Kk}{}_{;K} + \rho_0 f^k - \rho_0 \frac{D^2 u^k}{Dt^2} \right) \delta u_k \,\mathrm{d}V = 0. \quad (1.330)$$

So far we only specified $\delta \boldsymbol{u}$ to be a virtual displacement field satisfying the geometric boundary conditions. Now, we require Equation (1.330) to be valid not only for one special $\delta \boldsymbol{u}$ but also for any $\delta \boldsymbol{u}$ satisfying $\delta \boldsymbol{u} = 0$ on A_{0u}. Because of the arbitrariness of $\delta \boldsymbol{u}$, the functional analysis density theorem applies (for a proof, the reader is referred to (Belytschko *et al.* 2000)) requiring the coefficients of δu_k in each term in Equation (1.330) to be zero. Accordingly,

$$T^k_{(\boldsymbol{N})} - \overline{T}^k_{(\boldsymbol{N})} = 0 \quad \text{on } A_{0t} \quad (1.331)$$

$$T^{k+}_{(\boldsymbol{N}^+)} + T^{k-}_{(\boldsymbol{N}^-)} \quad \text{on } A_{0i} \quad (1.332)$$

$$P^{Kk}{}_{;K} + \rho_0 f^k = \rho_0 \frac{D^2 u^k}{Dt^2} \quad \text{on } V_0 \quad (1.333)$$

which is the strong form.

1.11.4 The weak form in spatial coordinates

Here again, the starting point is the strong form, Equation (1.281)

$$(\sigma^{kl} \boldsymbol{g}_l)_{,k} + \rho f^l \boldsymbol{g}_l = \rho \frac{Dv^l}{Dt} \boldsymbol{g}_l \quad (1.334)$$

subject to

$$t^{k+}_{(\boldsymbol{n}^+)} + t^{k-}_{(\boldsymbol{n}^-)} \quad \text{on } A_i \quad (1.335)$$

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \quad \text{on } A_u \quad (1.336)$$

$$t^k_{(\boldsymbol{n})} - \overline{t}^k_{(\boldsymbol{n})} = 0 \quad \text{on } A_t. \quad (1.337)$$

In spatial coordinates, a virtual velocity field $\delta \boldsymbol{v}$ is selected satisfying $\delta \boldsymbol{v} = 0$ on A_u. The reason for this will become clear in the derivation. Scalar multiplication yields

$$\int_V \left[(\sigma^{kl} \boldsymbol{g}_l)_{,k} \cdot \delta \boldsymbol{v} + \rho f^l \boldsymbol{g}_l \cdot \delta \boldsymbol{v} \right] \mathrm{d}v = \int_V \rho \frac{Dv^l}{Dt} \boldsymbol{g}_l \cdot \delta \boldsymbol{v} \, \mathrm{d}v. \tag{1.338}$$

Since

$$(\sigma^{kl} \boldsymbol{g}_l)_{,k} \cdot \delta \boldsymbol{v} = (\sigma^{kl} \boldsymbol{g}_l \cdot \delta \boldsymbol{v})_{,k} - \sigma^{kl} \boldsymbol{g}_l \cdot \delta \boldsymbol{v}_{,k} \tag{1.339}$$

and

$$\delta \boldsymbol{v}_{,k} = (\delta v_m \boldsymbol{g}^m)_{,k} = \delta v_{m;k} \boldsymbol{g}^m \tag{1.340}$$

one obtains

$$\int_V \left[(\sigma^{kl} \delta v_l)_{,k} - \sigma^{kl} \delta v_{l;k} + \rho \left(f^l - \frac{Dv^l}{Dt} \right) \delta v_l \right] \mathrm{d}v = 0 \tag{1.341}$$

or

$$\int_A \sigma^{kl} n_k \delta v_l \, \mathrm{d}a - \int_V \left[\sigma^{kl} \delta d_{kl} - \rho \left(f^l - \frac{Dv^l}{Dt} \right) \delta v_l \right] \mathrm{d}v = 0. \tag{1.342}$$

Indeed, $d_{kl} = (v_{k;l} + v_{l;k})/2$ and $\sigma^{kl} = \sigma^{lk}$. Taking into account the boundary conditions finally yields

$$\int_V \sigma^{kl} \delta d_{kl} \, \mathrm{d}v = \int_{A_t} \bar{t}^l_{(\boldsymbol{n})} \delta v_l \, \mathrm{d}a + \int_V \rho f^l \delta v_l \, \mathrm{d}v - \int_V \rho \frac{Dv^l}{Dt} \delta v_l \, \mathrm{d}v. \tag{1.343}$$

Equation (1.53) expresses the principle of virtual power. Notice that the principle of virtual work is of no avail here since the expression $\sigma^{kl} \delta u_{l;k}$ cannot be simplified because of the presence of nonlinear terms in the definition of the Eulerian strain measure. Accordingly, the spatial description implies a rate formulation and necessitates a thorough discussion of objective rate tensors. This can be largely avoided by using the material description.

Naturally, the strong form can also be obtained starting from the weak form. Interested readers are referred to (Belytschko *et al.* 2000).

1.12 The Weak Form of the Energy Balance

We start from the strong form expressed by Equation (1.298)

$$\rho_0 \frac{D\varepsilon}{Dt} = \dot{\boldsymbol{E}} : \boldsymbol{S} - \nabla_0 \cdot \boldsymbol{Q} + \rho_0 h \quad \text{on } V \tag{1.344}$$

completed by appropriate boundary conditions: we assume that the temperature T and the flux $\boldsymbol{Q}$ are known on A_{0T} and on A_{0Q} respectively. Furthermore, the flux normal to an interface A_i is continuous.

$$T = \overline{T} \quad \text{on } A_{0T} \tag{1.345}$$

$$\boldsymbol{Q} = \overline{\boldsymbol{Q}} \quad \text{on } A_{0Q} \tag{1.346}$$

$$\boldsymbol{Q}^+ \cdot \boldsymbol{N}^+ + \boldsymbol{Q}^- \cdot \boldsymbol{N}^- = 0 \quad \text{on } A_{0i}. \tag{1.347}$$

To obtain the weak form, we consider again an infinitesimal perturbation of the independent variable field T, satisfying the "geometric" boundary conditions in Equation (1.345). Hence,

$$\delta T = 0 \quad \text{on } A_{0T}. \tag{1.348}$$

Multiplying Equation (1.343) with δT and integrating over V, one obtains

$$\int_{V_0} \rho_0 \frac{D\varepsilon}{Dt} \delta T \, \mathrm{d}V = \int_{V_0} (\dot{\boldsymbol{E}} : \boldsymbol{S} - \nabla_0 \cdot \boldsymbol{Q} + \rho_0 h) \delta T \, \mathrm{d}V. \tag{1.349}$$

The second term on the right can be written as

$$\begin{aligned} -\int_{V_0} \nabla_0 \cdot \boldsymbol{Q} \delta T \, \mathrm{d}V &= -\int_{V_0} Q^K{}_{,K} \delta T \, \mathrm{d}V \\ &= -\int_{V_0} \left[(Q^K \delta T)_{,K} - Q^K \delta T_{,K} \right] \mathrm{d}V \\ &= -\int_{A_{0Q}} \delta T \, Q^K N_K \, \mathrm{d}A + \int_{V_0} Q^K \delta T_{,K} \, \mathrm{d}V. \end{aligned} \tag{1.350}$$

This step is essential to reduce the degree of differentiation of T in the resulting equation and is similar to Equations (1.318) to (1.320) for the balance of momentum. Indeed, the constitutive equations in Section 1.13 will show that $\boldsymbol{Q} \sim -\nabla_0 T$. Consequently, $\nabla_0 \cdot \boldsymbol{Q} \sim -\nabla_0 \cdot \nabla_0 T = -\nabla_0^2 T$ and $\nabla_0 \cdot \boldsymbol{Q} \; \delta T$ is the product of two terms of which the first one is twice differentiated, the second one not at all. On the other hand, both terms in $Q^K \delta T_{,K}$ are differentiated only once. This implies that the shape functions in the finite element formulation can have a lesser degree of smoothness and still comply with Equation (1.350).

Substitution of Equation (1.350) into Equation (1.349) yields

$$-\int_{V_0} \boldsymbol{Q} \cdot \delta \nabla_0 T \, \mathrm{d}V = \int_{V_0} \dot{\boldsymbol{E}} : \boldsymbol{S} \delta T \, \mathrm{d}V - \int_{A_{0Q}} \boldsymbol{Q} \cdot \boldsymbol{N} \delta T \, \mathrm{d}A + \int_{V_0} \rho_0 \left(h - \frac{D\varepsilon}{Dt} \right) \delta T \, \mathrm{d}V. \tag{1.351}$$

This equation is the analogue of Equation (1.328). Similar to what was said in Section 1.11.3, the strong form can be derived from the weak form if one allows the temperature perturbation to be absolutely general provided the "geometric" boundary conditions are satisfied.

1.13 Constitutive Equations

1.13.1 Summary of the balance equations

In Section 1.10, the balance equations were derived in material coordinates. They amount to

$$\rho J = \rho_0 \quad \text{(1 equation)} \tag{1.352}$$

$$\nabla_0 (\boldsymbol{S} \cdot \boldsymbol{F}^{\mathrm{T}}) + \rho_0 \boldsymbol{f} = \rho_0 \dot{\boldsymbol{v}} \quad \text{(3 equations)} \tag{1.353}$$

$$\boldsymbol{S} = \boldsymbol{S}^{\mathrm{T}} \quad \text{(3 equations)} \tag{1.354}$$

$$\rho_0 \dot{\varepsilon} = \dot{\boldsymbol{E}} : \boldsymbol{S} - \nabla_0 \cdot \boldsymbol{Q} + \rho_0 h \quad \text{(1 equation)} \tag{1.355}$$

$$\rho_0 \left(\dot{\eta} - \frac{\dot{\varepsilon}}{\theta} \right) + \frac{1}{\theta} \dot{\boldsymbol{E}} : \boldsymbol{S} - \frac{1}{\theta^2} \boldsymbol{Q} \cdot (\nabla_0 \theta) \geq 0. \tag{1.356}$$

In sum, there are eight equations and one inequality. The unknowns are ρ (1), J (1), $\boldsymbol{S}$ (9), $\boldsymbol{F}$ (9), $\boldsymbol{v}$ (3), ε (1), $\boldsymbol{E}$(6), $\boldsymbol{Q}$ (3), η (1) and θ (1) which yields 35 unknowns. The variables J, $\boldsymbol{F}$, $\boldsymbol{v}$ and $\boldsymbol{E}$ can be reduced to $\boldsymbol{x}$ (3 unknowns) since

$$J = \det(x^k_{,K}) \tag{1.357}$$

$$\boldsymbol{F} = x^k_{,K} \boldsymbol{g}_k \otimes \boldsymbol{G}^K \tag{1.358}$$

$$\boldsymbol{v} = \dot{\boldsymbol{x}} \tag{1.359}$$

$$\boldsymbol{E} = \tfrac{1}{2}(x^k_{,K} x^l_{,L} g_{kl} - G_{KL}) \boldsymbol{G}^K \otimes \boldsymbol{G}^L. \tag{1.360}$$

In that way, 19 unknowns remain. Accordingly, we need another 11 equations to solve the problem for $\boldsymbol{x}(\boldsymbol{X}, t)$ and $\theta(\boldsymbol{X}, t)$. This is not surprising, since the material properties were not considered so far. All balance equations apply to steel as well as to wood, water or air. It is well known that these materials behave quite differently and it is the task of the constitutive equations to describe these different kinds of behavior. It looks like a huge task to tackle but luckily there are some physical principles that may guide us. Here I wish to adhere to a simplified form of the axiomatic formulation found in (Eringen 1980) since it leads us in a systematic way to the constitutive equations of widely different materials.

1.13.2 Development of the constitutive theory

The constitutive equations bridge the gap between physically observable quantities (independent variables in the constitutive equations) and the quantities arising in the balance laws (dependent variables in the constitutive equations). For thermomechanical processes, the observable quantities are the location $\boldsymbol{x}(\boldsymbol{X}, t)$ and the temperature $\theta(\boldsymbol{X}, t)$. All other variables such as the stress $\boldsymbol{S}$, the flux $\boldsymbol{Q}$, the internal energy ε and the entropy η are measured indirectly by the effect they produce on the displacements and the temperature. For instance, the stress is usually measured through strain gauges. Accordingly, the value of the dependent variables ($\boldsymbol{S}$, $\boldsymbol{Q}$, ε, and η – the density ρ is not considered as a dependent variable but rather immediately eliminated through Equation (1.352)) at $\boldsymbol{X}$ at time t is assumed to be a function of the value of the independent variables ($\boldsymbol{x}$, θ) at all former times and in the complete body. This can be written in the form of the following functionals:

$$\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}[\boldsymbol{x}(\boldsymbol{X}', t'), \theta(\boldsymbol{X}', t'), \boldsymbol{X}, t] \tag{1.361}$$

$$\boldsymbol{Q}(\boldsymbol{X}, t) = \boldsymbol{Q}[\boldsymbol{x}(\boldsymbol{X}', t'), \theta(\boldsymbol{X}', t'), \boldsymbol{X}, t] \tag{1.362}$$

$$\varepsilon(\boldsymbol{X}, t) = \varepsilon[\boldsymbol{x}(\boldsymbol{X}', t'), \theta(\boldsymbol{X}', t'), \boldsymbol{X}, t] \tag{1.363}$$

$$\eta(\boldsymbol{X}, t) = \eta[\boldsymbol{x}(\boldsymbol{X}', t'), \theta(\boldsymbol{X}', t'), \boldsymbol{X}, t] \tag{1.364}$$

$$t' \leq t, \boldsymbol{X}' \in V_0.$$

Hence, *a priori* it is assumed that the deformation and temperature in the complete body at all former times can have an impact on the value of any dependent variable at

$\boldsymbol{X}$ and t. This formulation includes memory effects (e.g. viscosity) and nonlocal effects (atomic forces).

There are two major postulates that must be obeyed by the constitutive equations. First, there is the principle of objectivity, which states that the constitutive equations must not depend on the spatial motion of the observer. This principle has already been briefly discussed in Section 1.6. There, it was emphasized that only objective tensors should be used in constitutive equations. What does this translate to in Equations (1.361) to (1.364)? Since the left-hand side of these equations is formulated in terms of material quantities, objectivity is no problem. What about the right-hand side? In general, a time-dependent rigid body motion combined with a time-shift maps $\boldsymbol{x}(\boldsymbol{X}, t)$ into

$$\overline{\boldsymbol{x}}(\boldsymbol{X}', \overline{t'}) = \boldsymbol{Q}(t') \cdot \boldsymbol{x}(\boldsymbol{X}', t') + \boldsymbol{b}(t') \tag{1.365}$$

$$\boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{Q} = \boldsymbol{I}, \quad \overline{t'} = t' - a. \tag{1.366}$$

This mapping can be split into a time-dependent translation, a time-shift and a time-dependent rotation.

A time-dependent translation must not change the constitutive equation. Taking the translation to be $\boldsymbol{x}(\boldsymbol{X}, t')$ one obtains, Equation (1.361),

$$\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}[\boldsymbol{x}(\boldsymbol{X}', t') - \boldsymbol{x}(\boldsymbol{X}, t'), \theta(\boldsymbol{X}', t'), \boldsymbol{X}, t], \tag{1.367}$$

which means that only the relative position with respect to $\boldsymbol{x}(\boldsymbol{X}, t')$ is kept.
A time shift must not influence Equation (1.367) either. Taking the shift to be t one obtains

$$\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}[\boldsymbol{x}(\boldsymbol{X}', t' - t) - \boldsymbol{x}(\boldsymbol{X}, t' - t), \theta(\boldsymbol{X}', t' - t), \boldsymbol{X}, 0]. \tag{1.368}$$

Consequently, the explicit dependence on t drops out:

$$\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}[\boldsymbol{x}(\boldsymbol{X}', t') - \boldsymbol{x}(\boldsymbol{X}, t'), \theta(\boldsymbol{X}', t'), \boldsymbol{X}]. \tag{1.369}$$

Finally, a time-dependent rotation of the spatial frame of reference should also leave the constitutive equation unaltered. One obtains

$$\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}\left[\boldsymbol{Q}(t) \cdot (\boldsymbol{x}(\boldsymbol{X}', t') - \boldsymbol{x}(\boldsymbol{X}, t')), \theta(\boldsymbol{X}', t'), \boldsymbol{X}\right] \tag{1.370}$$

for an arbitrary rotation $\boldsymbol{Q}(t)$.

The second postulate states that the constitutive equations must be form-invariant with respect to a certain class of rotations $\boldsymbol{Q}$ and translations $\boldsymbol{B}$ of the material frame, which are the result of material symmetries and material homogeneities. A lot of materials exhibit symmetries with respect to a specific class of rotations due to the intrinsic crystallographic structure. For example, single crystals are frequently orthotropic, which means that mutually orthogonal planes exist in the material frame with respect to which the material properties are symmetric. A usual assumption in polycrystals is the form-invariance with respect to the full group of rotations: this means that the material properties are independent of the direction and this is called *isotropy*. For a homogeneous material, the properties are not changed by an arbitrary translation. Once the classes $\{\boldsymbol{Q}\}$ and $\{\boldsymbol{B}\}$ are determined on the basis of observations of the material behavior, the constitutive equations should be form-invariant with respect to transformations of the type

$$\overline{\boldsymbol{X}} = \boldsymbol{Q} \cdot \boldsymbol{X} + \boldsymbol{B}, \quad \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathrm{T}} = \boldsymbol{I}, \quad \det \boldsymbol{Q} = \pm 1 \tag{1.371}$$

mapping the material frame $\boldsymbol{X}$ into $\overline{\boldsymbol{X}}$. Notice that the axiom of objectivity involves transformations of the spatial frame, whereas the axiom of material invariance concerns transformations of the material frame.

In order to obtain further simplifications, the expressions for the independent quantities are expanded in a Taylor series. Taylor expansion of $\boldsymbol{x}(\boldsymbol{X}', t') - \boldsymbol{x}(\boldsymbol{X}, t')$ in space yields

$$\begin{aligned}\boldsymbol{x}(\boldsymbol{X}', t') - \boldsymbol{x}(\boldsymbol{X}, t') &= \boldsymbol{x}_{,K_1}(\boldsymbol{X}, t')\left(X'^{K_1} - X^{K_1}\right) \\ &\quad + \frac{1}{2!}\boldsymbol{x}_{,K_1K_2}(\boldsymbol{X}, t')\left(X'^{K_1} - X^{K_1}\right)\left(X'^{K_2} - X^{K_2}\right) + \cdots\end{aligned} \tag{1.372}$$

Similarly,

$$\begin{aligned}\theta(\boldsymbol{X}', t') &= \theta(\boldsymbol{X}, t') + \theta_{,K_1}(\boldsymbol{X}, t')\left(X'^{K_1} - X^{K_1}\right) \\ &\quad + \frac{1}{2!}\theta_{,K_1K_2}(\boldsymbol{X}, t')\left(X'^{K_1} - X^{K_1}\right)\left(X'^{K_2} - X^{K_2}\right) + \cdots\end{aligned} \tag{1.373}$$

and Equation (1.98) can be replaced by

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}[&\boldsymbol{Q}(t) \cdot \boldsymbol{x}_{,K_1}(\boldsymbol{X}, t'),\ \boldsymbol{Q}(t) \cdot \boldsymbol{x}_{,K_1K_2}(\boldsymbol{X}, t'), \dots, \\ &\theta(\boldsymbol{X}, t'), \theta_{,K_1}(\boldsymbol{X}, t'), \theta_{,K_1K_2}(\boldsymbol{X}, t'), \dots, \boldsymbol{X}].\end{aligned} \tag{1.374}$$

Notice that the dependent variables are explicitly dependent on $\theta(\boldsymbol{X}, t')$ but not on $\boldsymbol{x}(\boldsymbol{X}, t')$. Materials satisfying Equation (1.374) are said to be of mechanical grade N and thermal grade M if the spatial derivatives are at most of Nth order and the thermal derivatives of at most Mth order.

Taylor expanding the remaining independent variables in time yields

$$\boldsymbol{x}_{,K_1}(\boldsymbol{X}, t') = \boldsymbol{x}_{,K_1}(\boldsymbol{X}, t) + \dot{\boldsymbol{x}}_{,K_1}(\boldsymbol{X}, t)(t' - t) + \frac{1}{2!}\ddot{\boldsymbol{x}}_{,K_1}(\boldsymbol{X}, t)(t' - t)^2 + \cdots \tag{1.375}$$

and similar for the other variables. Hence, one can replace Equation (1.374) by

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}[&\boldsymbol{Q}(t) \cdot \boldsymbol{x}_{,K_1}(\boldsymbol{X}, t),\ \boldsymbol{Q}(t) \cdot \dot{\boldsymbol{x}}_{,K_1}(\boldsymbol{X}, t), \dots, \\ &\boldsymbol{Q}(t) \cdot \boldsymbol{x}_{,K_1K_2}(\boldsymbol{X}, t),\ \boldsymbol{Q}(t) \cdot \dot{\boldsymbol{x}}_{,K_1K_2}(\boldsymbol{X}, t), \dots, \\ &\dots \\ &\boldsymbol{Q}(t) \cdot \boldsymbol{x}_{,K_1K_2\dots K_N}(\boldsymbol{X}, t),\ \boldsymbol{Q}(t) \cdot \dot{\boldsymbol{x}}_{,K_1K_2\dots K_N}(\boldsymbol{X}, t), \dots, \\ &\theta(\boldsymbol{X}, t), \dot{\theta}(\boldsymbol{X}, t), \ddot{\theta}(\boldsymbol{X}, t), \dots, \\ &\theta_{,K_1}(\boldsymbol{X}, t), \dot{\theta}_{,K_1}(\boldsymbol{X}, t), \ddot{\theta}_{,K_1}(\boldsymbol{X}, t), \dots, \\ &\dots \\ &\theta_{,K_1K_2\dots K_M}(\boldsymbol{X}, t), \dot{\theta}_{,K_1K_2\dots K_M}(\boldsymbol{X}, t), \ddot{\theta}_{,K_1K_2\dots K_M}(\boldsymbol{X}, t), \dots, \boldsymbol{X}].\end{aligned} \tag{1.376}$$

In what follows, we will concentrate on materials of mechanical grade 1 and thermal grade 1. Hence,

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{X},t) = \boldsymbol{S}[&\boldsymbol{Q}(t)\cdot \boldsymbol{x}_{,K}(\boldsymbol{X},t),\ \boldsymbol{Q}(t)\cdot \dot{\boldsymbol{x}}_{,K}(\boldsymbol{X},t),\dots,\\ &\theta(\boldsymbol{X},t),\dot{\theta}(\boldsymbol{X},t),\ddot{\theta}(\boldsymbol{X},t),\dots,\\ &\theta_{,K}(\boldsymbol{X},t),\dot{\theta}_{,K}(\boldsymbol{X},t),\ddot{\theta}_{,K}(\boldsymbol{X},t),\dots,\boldsymbol{X}].\end{aligned} \tag{1.377}$$

The principle of objectivity implies that the right-hand side of Equation (1.377) must be invariant with respect to spatial rotations. This means that the list of independent variables can be replaced by the invariants of $\{\boldsymbol{x}_{,K}, \dot{\boldsymbol{x}}_{,K}, \ddot{\boldsymbol{x}}_{,K}, \dots\}$ with respect to an arbitrary rotation. The theory of invariants (Spencer 1971) shows that an integrity basis for the invariants of the above set subject to proper transformations (i.e. det $\boldsymbol{Q} = +1$) consists of the scalar product of any two vectors in the set, for example,

$$\boldsymbol{x}_{,K}\cdot \boldsymbol{x}_{,L} = x^k_{,K}x^l_{,L}g_{kl} = C_{KL} \tag{1.378}$$

and triple products of the form

$$e_{klm}x^k_{,K}x^l_{,L}x^m_{,M}. \tag{1.379}$$

For $K \neq L$, $K \neq M$ and $L \neq M$ the expression in Equation (1.379) is the Jacobian determinant J. For $K = L$, $K = M$ or $K = M$ Equation (1.379) is zero since this amounts to the determinant of a matrix with two equal rows or columns. Consequently, the dependence on $\{\boldsymbol{x}_{,K}, \dot{\boldsymbol{x}}_{,K}, \ddot{\boldsymbol{x}}_{,K}, \dots\}$ in Equation (1.377) can be replaced by a dependence on

$$\{C_{KL}, \dot{C}_{KL}, \ddot{C}_{KL}, \dots, J, \dot{J}, \ddot{J}, \dots\} \tag{1.380}$$

or

$$\{C_{KL}, \dot{C}_{KL}, \ddot{C}_{KL}, \dots, \rho^{-1}, \dot{\rho}, \ddot{\rho}, \dots\} \tag{1.381}$$

since $J = \rho_0\rho^{-1}$. Equation (1.377) now reads

$$\begin{aligned}\boldsymbol{S}(\boldsymbol{X},t) = \boldsymbol{S}[&C_{KL}(\boldsymbol{X},t), \dot{C}_{KL}(\boldsymbol{X},t),\dots,\\ &\rho^{-1}(\boldsymbol{X},t), \dot{\rho}(\boldsymbol{X},t),\dots,\\ &\theta(\boldsymbol{X},t),\dot{\theta}(\boldsymbol{X},t),\ddot{\theta}(\boldsymbol{X},t),\cdots,\\ &\theta_{,K}(\boldsymbol{X},t),\dot{\theta}_{,K}(\boldsymbol{X},t),\ddot{\theta}_{,K}(\boldsymbol{X},t),\dots,\boldsymbol{X}].\end{aligned} \tag{1.382}$$

This also applies to $\boldsymbol{Q}$, ε and η yielding the missing 11 equations.

1.14 Elastic Materials

1.14.1 General form

Elastic materials are defined as *materials without memory*. Consequently, the time derivatives are dropped in Equation (1.382) and one obtains

$$\boldsymbol{S}(\boldsymbol{X},t) = \boldsymbol{S}[\boldsymbol{C}(\boldsymbol{X},t), \rho^{-1}(\boldsymbol{X},t), \theta(\boldsymbol{X},t), \nabla_0\theta(\boldsymbol{X},t), \boldsymbol{X}] \tag{1.383}$$

and similarly (dropping the dependence),

$$\boldsymbol{Q} = \boldsymbol{Q}[\boldsymbol{C}, \rho^{-1}, \theta, \nabla_0\theta, \boldsymbol{X}] \tag{1.384}$$

$$\varepsilon = \varepsilon[\boldsymbol{C}, \rho^{-1}, \theta, \nabla_0\theta, \boldsymbol{X}] \tag{1.385}$$

$$\eta = \eta[\boldsymbol{C}, \rho^{-1}, \theta, \nabla_0\theta, \boldsymbol{X}]. \tag{1.386}$$

Since $\det \boldsymbol{C} = J^2$ and $\rho J = \rho_0$ (balance of mass) the explicit dependence on ρ is dropped. The balance of momentum, the balance of energy and the entropy inequality remain to be satisfied. It is amazing that the entropy inequality, being an inequality, plays an extremely important role in the derivation of the material laws. To see this, we first define the free energy $\psi(\boldsymbol{X}, t)$ to simplify the calculations:

$$\psi := \varepsilon - \theta\eta. \tag{1.387}$$

Since

$$\dot{\psi} = \dot{\varepsilon} - \dot{\theta}\eta - \theta\dot{\eta} \tag{1.388}$$

the entropy inequality now reads

$$-\frac{\rho_0}{\theta}(\dot{\psi} + \dot{\theta}\eta) + \frac{1}{\theta}\boldsymbol{S} : \dot{\boldsymbol{E}} - \frac{1}{\theta^2}\boldsymbol{Q} \cdot \nabla_0\theta \geq 0. \tag{1.389}$$

Notice that because of Equation (1.385) and Equation (1.386), dropping the dependence on ρ^{-1}

$$\psi = \psi[\boldsymbol{C}, \theta, \nabla_0\theta, \boldsymbol{X}]. \tag{1.390}$$

Accordingly,

$$\dot{\psi} = \frac{\partial\psi}{\partial\boldsymbol{C}} : \dot{\boldsymbol{C}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\nabla_0\theta} \cdot \dot{\overline{\nabla_0\theta}}. \tag{1.391}$$

Substituting Equation (1.391) into Equation (1.389) and noting that $\dot{\boldsymbol{E}} = \dot{\boldsymbol{C}}/2$ yields

$$\frac{1}{\theta}\left(-\rho_0\frac{\partial\psi}{\partial\boldsymbol{C}} + \tfrac{1}{2}\boldsymbol{S}\right) : \dot{\boldsymbol{C}} - \frac{\rho_0}{\theta}\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} - \frac{\rho_0}{\theta}\frac{\partial\psi}{\partial\nabla_0\theta} \cdot \dot{\overline{\nabla_0\theta}} - \frac{1}{\theta^2}\boldsymbol{Q} \cdot \nabla_0\theta \geq 0. \tag{1.392}$$

Since $\boldsymbol{S}$, $\boldsymbol{Q}$, ψ and η are not a function of $\dot{\boldsymbol{C}}$ nor $\dot{\theta}$ nor $\dot{\overline{\nabla_0\theta}}$, Equation (1.392) is linear in $\dot{\boldsymbol{C}}$, $\dot{\theta}$ and $\nabla_0\dot{\theta}$. Hence, for Equation (1.392) to be valid for any $\dot{\boldsymbol{C}}$, $\dot{\theta}$ or $\nabla_0\dot{\theta}$, the coefficients of these terms must be zero. Defining $\Sigma := \rho_0\psi$ one obtains

$$\boldsymbol{S} = 2\rho_0\frac{\partial\psi}{\partial\boldsymbol{C}} = 2\frac{\partial\Sigma}{\partial\boldsymbol{C}} = \frac{\partial\Sigma}{\partial\boldsymbol{E}} \tag{1.393}$$

$$\eta = -\frac{\partial\psi}{\partial\theta} = -\frac{1}{\rho_0}\frac{\partial\Sigma}{\partial\theta} \tag{1.394}$$

$$\frac{\partial\psi}{\partial\nabla_0\theta} = \frac{\partial\Sigma}{\partial\nabla_0\theta} = 0 \tag{1.395}$$

and the entropy inequality reduces to

$$-\boldsymbol{Q} \cdot \nabla_0 \theta \geq 0. \tag{1.396}$$

Consequently, for elastic materials there exists a function $\Sigma(\boldsymbol{C}, \theta, \boldsymbol{X})$ such that $\boldsymbol{S}$ and η can be obtained by partial differentiation. ε satisfies

$$\varepsilon(\boldsymbol{C}, \theta, \boldsymbol{X}) = \frac{\Sigma}{\rho_0} + \theta\eta. \tag{1.397}$$

The only dependent variable that depends on $\nabla_0\theta$ is $\boldsymbol{Q}$. Equation (1.396) requires that $\boldsymbol{Q}$ is at least linear in $\nabla_0\theta$, that is,

$$\boldsymbol{Q} = -\kappa(\boldsymbol{C}, \theta, \nabla_0\theta, \boldsymbol{X}) \cdot \nabla_0\theta. \tag{1.398}$$

The entropy inequality has dictated the shape of nearly all variables! The only equations left to satisfy are the balance of momentum and the balance of energy. Summarizing,

$$\boldsymbol{S} = \frac{\partial \Sigma}{\partial \boldsymbol{E}}(\boldsymbol{C}, \theta, \boldsymbol{X}) \tag{1.399}$$

$$\eta = -\frac{1}{\rho_0}\frac{\partial \Sigma}{\partial \theta}(\boldsymbol{C}, \theta, \boldsymbol{X}) \tag{1.400}$$

$$\varepsilon = \frac{\Sigma}{\rho_0} + \theta\eta \tag{1.401}$$

$$\boldsymbol{Q} = \boldsymbol{Q}(\boldsymbol{C}, \theta, \nabla_0\theta, \boldsymbol{X}). \tag{1.402}$$

Elastic materials in this general form are also called *hyperelastic materials*. Σ is sometimes called the *stored energy function* (Ciarlet 1993).

1.14.2 Linear elastic materials

Special forms arise if we linearize $\boldsymbol{S}$ with respect to $\boldsymbol{E}$ and $\boldsymbol{Q}$ with respect to $\boldsymbol{E}$ and $\nabla_0\theta$ ($\boldsymbol{C}$ and $\boldsymbol{E}$ are equivalent independent variables). To obtain a linear relation between $\boldsymbol{S}$ and $\boldsymbol{E}$, we expand Σ about $\boldsymbol{E} = 0$ and truncate the series after the quadratic terms:

$$\Sigma \sim \Sigma_0(\theta, \boldsymbol{X}) + \Sigma^{KL}(\theta, \boldsymbol{X})E_{KL} + \tfrac{1}{2}\Sigma^{KLMN}(\theta, \boldsymbol{X})E_{KL}E_{MN}, \quad \|\boldsymbol{E}\| \to 0 \tag{1.403}$$

while $\boldsymbol{Q}$ is expanded at $\nabla_0\theta = 0$, $\boldsymbol{E} = 0$ and the linear terms are kept

$$\boldsymbol{Q} \sim -\kappa^K(\theta, \boldsymbol{X}) - \kappa^{KL}(\theta, \boldsymbol{X})\theta_{,L} - \kappa^{KLM}(\theta, \boldsymbol{X})E_{LM}, \quad \|\nabla_0\theta\| \to 0, \|\boldsymbol{E}\| \to 0. \tag{1.404}$$

Because of the symmetry of $\boldsymbol{E}$ one finds

$$\Sigma^{KL} = \Sigma^{LK} \tag{1.405}$$

$$\Sigma^{KLMN} = \Sigma^{LKMN} = \Sigma^{KLNM} = \Sigma^{MNKL} \tag{1.406}$$

$$\kappa^{KLM} = \kappa^{KML}. \tag{1.407}$$

Applying Equation (1.399) yields

$$S^{KL}(\theta, \boldsymbol{X}) = \Sigma^{KL}(\theta, \boldsymbol{X}) + \Sigma^{KLMN}(\theta, \boldsymbol{X})E_{MN}. \tag{1.408}$$

Physical observations and the second law of thermodynamics, cf Equations (1.396) and (1.398), dictate that there is no heat flux if the temperature gradient is zero. This leads to (see Equation (1.404))

$$\kappa^K = 0, \quad \kappa^{KLM} = 0 \tag{1.409}$$

and

$$Q^K = -\kappa^{KL}(\theta, \boldsymbol{X})\theta_{,L}. \tag{1.410}$$

The entropy inequality now amounts to

$$\kappa^{KL}\theta_{,K}\theta_{,L} \geq 0 \tag{1.411}$$

which means that the symmetric part of κ^{KL} must be positive definite. The physical meaning of κ^{KL} is the conduction coefficient matrix.

The term $\Sigma^{KL}(\theta, \boldsymbol{X})$ in Equation (1.408) contains the thermal stress. Let the temperature θ_{ref} represent a homogeneous temperature distribution without any thermal stresses. Then one can write

$$\Sigma^{KL}(\theta, \boldsymbol{X}) = \gamma^{KL}(\boldsymbol{X}) - \beta^{KL}(\theta, \boldsymbol{X})(\theta - \theta_{\text{ref}}). \tag{1.412}$$

γ^{KL} are residual stresses from other sources and β^{kl} is the compressive stress rise per unit temperature increase if no expansion is allowed. Furthermore, we define α^{KL} assuming Σ^{KLMN} to be invertible:

$$\beta^{KL}(\theta, \boldsymbol{X}) =: \Sigma^{KLMN}(\theta, \boldsymbol{X})\alpha_{MN}(\theta, \boldsymbol{X}). \tag{1.413}$$

Hence,

$$\begin{aligned} S^{KL}(\theta, \boldsymbol{X}) &= \gamma^{KL}(\boldsymbol{X}) - \beta^{KL}(\theta, \boldsymbol{X})(\theta - \theta_{\text{ref}}) + \Sigma^{KLMN}(\theta, \boldsymbol{X})E_{MN} \\ &= \gamma^{KL}(\boldsymbol{X}) + \Sigma^{KLMN}(\theta, \boldsymbol{X})\left[E_{MN} - \alpha_{MN}(\theta, \boldsymbol{X})(\theta - \theta_{\text{ref}})\right]. \end{aligned} \tag{1.414}$$

The tensor $\boldsymbol{\alpha}$ contains the expansion coefficients. Now, let us expand $\Sigma_0(\theta, \boldsymbol{X})$ in Equation (1.403) about θ_{ref}:

$$\Sigma_0(\theta, \boldsymbol{X}) = \rho_0(\boldsymbol{X})\psi_0(\boldsymbol{X}) - \rho_0(\boldsymbol{X})\eta_0(\boldsymbol{X})(\theta - \theta_{\text{ref}}) - \frac{\rho_0(\boldsymbol{X})c(\theta, \boldsymbol{X})}{2\theta_{\text{ref}}}(\theta - \theta_{\text{ref}})^2. \tag{1.415}$$

Notice that the equality sign applies, since $c(\theta, \boldsymbol{X})$ in the last term may depend on θ. Dropping the dependence on $\boldsymbol{X}$ in the notation and defining $T := \theta - \theta_{\text{ref}}$ yields

$$\Sigma = \rho_0\psi_0 - \rho_0\eta_0 T - \frac{\rho_0 c(\theta)}{2\theta_{\text{ref}}}T^2 + [\gamma^{KL} - \beta^{KL}(\theta)T]E_{KL} + \frac{1}{2}\Sigma^{KLMN}(\theta)E_{KL}E_{MN} \tag{1.416}$$

and

$$\eta = \eta_0 + \frac{c(\theta)T}{\theta_{\text{ref}}} + \frac{\beta^{KL}(\theta)}{\rho_0}E_{KL} + \frac{c'(\theta)T^2}{2\theta_{\text{ref}}} + \frac{\beta^{KL'}(\theta)}{\rho_0}TE_{KL} + \frac{1}{2}\Sigma^{KLMN'}(\theta)E_{KL}E_{MN}. \tag{1.417}$$

The last three terms are due to the temperature dependence of the coefficients. Since $\rho_0\varepsilon = \Sigma + \rho_0\theta\eta$, one obtains

$$\begin{aligned}\rho_0\varepsilon = \rho_0\psi_0 + \rho_0\eta_0\theta_{\text{ref}} + \rho_0 c(\theta)\left(T + \frac{T^2}{2\theta_{\text{ref}}}\right) \\ + [\gamma^{KL} + \beta^{KL}(\theta)\theta_{\text{ref}}]E_{KL} + \frac{1}{2}\Sigma^{KLMN}(\theta)E_{KL}E_{MN} + \\ \rho_0\frac{\theta c'(\theta)T^2}{2\theta_{\text{ref}}} + \theta\beta^{KL'}(\theta)TE_{KL} + \frac{1}{2}\rho_0\theta\Sigma^{KLMN'}(\theta)E_{KL}E_{MN}.\end{aligned} \tag{1.418}$$

From Equation (1.418), it follows that c is the specific heat. Substituting the above equations into the energy balance, Equation (1.355), is quite a tedious task. Generally, the derivative of the coefficients with respect to the temperature can be neglected (the coefficients, however, are still a function of temperature). Furthermore, discarding the quadratic T term leads to

$$\rho_0\varepsilon = \rho_0\psi_0 + \rho_0\eta_0\theta_{\text{ref}} + \rho_0c(\theta)T + [\gamma^{KL} + \beta^{KL}(\theta)\theta_{\text{ref}}]E_{KL} + \tfrac{1}{2}\Sigma^{KLMN}(\theta)E_{KL}E_{MN} \tag{1.419}$$

and for the stress

$$S^{KL} = [\gamma^{KL} - \beta^{KL}(\theta)T] + \Sigma^{KLMN}(\theta)E_{MN}. \tag{1.420}$$

Substitution into Equation (1.355) finally yields (after further linearization: $\theta\dot{E}_{KL} \approx \theta_{\text{ref}}\dot{E}_{KL}$)

$$\rho_0c(\theta)\dot{T} + \beta^{KL}(\theta)\theta_{\text{ref}}\dot{E}_{KL} - (\kappa^{KL}(\theta)\theta_{,L})_{;K} - \rho_0h = 0. \tag{1.421}$$

This is the classical heat equation for linear elastic materials.

If in Equation (1.420)

$$\begin{aligned}\gamma^{KL} &= 0 \quad \text{for } K \neq L \\ \beta^{KL} &= 0 \quad \text{for } K \neq L \\ \Sigma^{KLMN} &= 0 \quad \text{for } K \neq L \text{ and } M = N\end{aligned} \tag{1.422}$$

one obtains $S^{KL} = 0$, $K \neq L$ if $E_{KL} = 0$, $K \neq L$ and vice versa. If this is true for arbitrary orientations of the axes as in the case of isotropic materials, then the principal axes of $\boldsymbol{E}$ are also principal axes of $\boldsymbol{S}$. Indeed, take the principal axes of $\boldsymbol{E}$ as a local rectangular coordinate system. This means that $E_{KL} = 0$, $K \neq L$ and consequently $S^{KL} = 0$, $K \neq L$: the shear stress is zero. Accordingly,

$$\boldsymbol{E} = \sum_i \Lambda_{iE}\boldsymbol{N}^i \otimes \boldsymbol{N}^i \tag{1.423}$$

$$\boldsymbol{S} = \sum_i \Lambda_{iS}\boldsymbol{N}_i \otimes \boldsymbol{N}_i. \tag{1.424}$$

Furthermore, since

$$\boldsymbol{F} = \sum_i \sqrt{\Lambda_{iC}}\boldsymbol{n}_i \otimes \boldsymbol{N}^i \tag{1.425}$$

and

$$\boldsymbol{\sigma} = J^{-1}\boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^{\mathrm{T}} \tag{1.426}$$

one finds

$$\begin{aligned}\boldsymbol{\sigma} &= J^{-1}\left(\sum_i \sqrt{\Lambda_{iC}}\boldsymbol{n}_i \otimes \boldsymbol{N}^i\right) \cdot \left(\sum_j \Lambda_{jS}\boldsymbol{N}_j \otimes \boldsymbol{N}_j\right) \cdot \left(\sum_k \sqrt{\Lambda_{kC}}\boldsymbol{N}^k \otimes \boldsymbol{n}_k\right) \\ &= J^{-1}\sum_i \Lambda_{iC}\Lambda_{iS}\boldsymbol{n}_i \otimes \boldsymbol{n}_i \end{aligned} \tag{1.427}$$

which yields for the true principal stresses

$$\lambda_{i\sigma} = J^{-1}\Lambda_{\underline{i}C}\Lambda_{\underline{i}S} = J^{-1}(2\Lambda_{\underline{i}E} + 1)\Lambda_{\underline{i}S}. \tag{1.428}$$

Since J, $\Lambda_{Ci} > 0$, the true stress and the second Piola–Kirchhoff stress have the same sign.

1.14.3 Isotropic linear elastic materials

For a linear elastic material, we found

$$\Sigma = \rho_0\psi_0 - \rho_0\eta_0 T - \frac{\rho_0 c(\theta)}{2\theta_{\mathrm{ref}}}T^2 + [\gamma^{KL} - \beta^{KL}(\theta)T]E_{KL} + \frac{1}{2}\Sigma^{KLMN}(\theta)E_{KL}E_{MN} \tag{1.429}$$

$$S^{KL} = [\gamma^{KL} - \beta^{KL}(\theta)T] + \Sigma^{KLMN}(\theta)E_{MN} \tag{1.430}$$

$$Q^K = -\kappa^{KL}(\theta)\theta_{,L}. \tag{1.431}$$

Isotropy means that the material data are independent of the direction in the material frame of reference. Hence, a transformation $\boldsymbol{Q}$ such that

$$\boldsymbol{X}' = \boldsymbol{Q} \cdot \boldsymbol{X}, \quad \boldsymbol{Q}^{\mathrm{T}} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathrm{T}} = 1, \quad \det \boldsymbol{Q} = \pm 1 \tag{1.432}$$

must leave the constitutive equations invariant. Under such a transformation, second-order and fourth-order tensors transform according to

$$\gamma'^{KL} = \gamma^{MN}Q^K{}_M Q^L{}_N \tag{1.433}$$

$$\Sigma'^{KLMN} = \Sigma^{PQRS}Q^K{}_P Q^L{}_Q Q^M{}_R Q^N{}_S. \tag{1.434}$$

One can show that for this to be true for an arbitrary rotation, the tensors must satisfy

$$\gamma^{KL} = \gamma G^{KL} \tag{1.435}$$

$$\Sigma^{KLMN} = \lambda G^{KL}G^{MN} + \mu(G^{KM}G^{LN} + G^{KN}G^{LM}) \tag{1.436}$$

and similarly for the other tensors

$$\beta^{KL} = \beta G^{KL} \tag{1.437}$$

$$\kappa^{KL} = \kappa G^{KL}. \tag{1.438}$$

Since

$$\mathrm{tr}\boldsymbol{E} = G^{KL} E_{KL} \tag{1.439}$$

is the trace of the tensor $\boldsymbol{E}$, one finds

$$\Sigma(\theta) = \rho_0 \psi_0 - \rho_0 \eta_0 T - \frac{\rho_0 c(\theta)}{2\theta_{\mathrm{ref}}} T^2 + [\gamma - \beta(\theta) T]\mathrm{tr}\boldsymbol{E} + \tfrac{1}{2}\lambda(\theta)(\mathrm{tr}\boldsymbol{E})^2 + \mu(\theta)\mathrm{tr}(\boldsymbol{E}^2) \tag{1.440}$$

$$S^{KL} = [\gamma - \beta(\theta)T]G^{KL} + \lambda(\theta)(\mathrm{tr}\boldsymbol{E})G^{KL} + 2\mu(\theta)E_{MN}G^{KM}G^{LN} \tag{1.441}$$

$$Q^K = -\kappa(\theta)\theta_{,L}G^{KL}. \tag{1.442}$$

The energy equation reduces to

$$\rho c(\theta)\dot{T} + \beta(\theta)\theta_{\mathrm{ref}}\dot{E}^K{}_K - (G^{KL}\kappa(\theta)\theta_{,L})_{;K} - \rho_0 h = 0. \tag{1.443}$$

The kind of material described by Equations (1.440) to (1.443) is also called a *St Venant–Kirchhoff material.*

The first and second invariant of a tensor $\boldsymbol{E}$ are defined by

$$I_{1E} = \mathrm{tr}\boldsymbol{E} \tag{1.444}$$

$$I_{2E} = \frac{1}{2}[I_{1E}^2 - \mathrm{tr}(\boldsymbol{E}^2)]. \tag{1.445}$$

Consequently, the free energy Σ can also be written as

$$\Sigma = \rho_0 \psi_0 - \rho_0 \eta_0 T - \frac{\rho_0 c(\theta)}{2\theta_{\mathrm{ref}}} T^2 + [\gamma - \beta(\theta)T]I_{1E} + \frac{1}{2}[\lambda(\theta) + 2\mu(\theta)]I_{1E}^2 - 2\mu(\theta)I_{2E}. \tag{1.446}$$

$\lambda(\theta)$ and $\mu(\theta)$ are called *Lamé's constants*, $\kappa(\theta)$ is the conduction coefficient, $c(\theta)$ is the specific heat and $\beta(\theta)$ satisfies (substitute Equation (1.436) into Equation (1.413))

$$\beta(\theta) = [3\lambda(\theta) + 2\mu(\theta)]\alpha(\theta) \tag{1.447}$$

where $\alpha(\theta)$ is the isotropic expansion coefficient. The thermal stress now yields

$$S^{KL} = -[3\lambda(\theta) + 2\mu(\theta)]\alpha(\theta)TG^{KL}. \tag{1.448}$$

This stress is needed to suppress

$$E_{KL} = \alpha(\theta)TG_{KL} \tag{1.449}$$

which is the strain resulting from the temperature change.

Finally, it should be noted that frequently other elastic constants are used instead of the Lamé's constants λ and μ, the latter of which is also called the *shear modulus*. The Poisson coefficient ν and Young's modulus E satisfy

$$\mu = \frac{E}{2(1+\nu)} \tag{1.450}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{1.451}$$

which can be inverted to yield

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \tag{1.452}$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}. \tag{1.453}$$

Another frequently used constant is the bulk modulus K. For linearized strains, it will be proven in the next section that K is the ratio of the hydrostatic pressure p to the volume reduction it produces. The following relations apply

$$K = \lambda + \frac{2}{3}\mu \tag{1.454}$$

$$\nu = \frac{3K-2\mu}{6K+2\mu}. \tag{1.455}$$

1.14.4 Linearizing the strains

So far, we consistently used the Lagrange strain tensor $\boldsymbol{E}$. Equation (1.82) shows that $\boldsymbol{E}$ is not linear in the displacement $\boldsymbol{U}$. To obtain a truly linear theory, the quadratic terms in $\boldsymbol{E}$ are dropped and one obtains the infinitesimal strains $\tilde{\boldsymbol{E}}$, Equation (1.88):

$$\tilde{E}_{KL} = \tfrac{1}{2}(U_{K;L} + U_{L;K}). \tag{1.456}$$

Recall that the infinitesimal rotation is defined by

$$\tilde{R}_{KL} = \tfrac{1}{2}(U_{K;L} - U_{L;K}). \tag{1.457}$$

Equation (1.90) has shown that E_{KL} can only be replaced by $\tilde{E}_{KL}$ if both the strain and the rotations are small. The same applies to e_{kl} and $\tilde{e}_{kl}$. Under the above assumptions, one can write

$$E_{KL} \sim \tilde{E}_{KL} \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0 \tag{1.458}$$

$$e_{kl} \sim \tilde{e}_{kl} \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.459}$$

To derive further simplifications we start from Equation (1.67):

$$x^k \boldsymbol{g}_k = (X^L + U^L)\boldsymbol{G}_L - \boldsymbol{o}. \tag{1.460}$$

Taking the derivative with respect to K yields

$$x^k_{,K}\boldsymbol{g}_k = (\delta^L{}_K + U^L{}_{;K})\boldsymbol{G}_L \tag{1.461}$$

leading to

$$x^k_{,K} = (\delta^L{}_K + \tilde{E}^L{}_K + \tilde{R}^L{}_K)\boldsymbol{G}_L \cdot \boldsymbol{g}^k \tag{1.462}$$

or

$$x^k_{,K} = (\delta^L{}_K + \tilde{E}^L{}_K + \tilde{R}^L{}_K)g_L{}^k \tag{1.463}$$

where

$$g_L{}^k = \boldsymbol{G}_L \cdot \boldsymbol{g}^k = \boldsymbol{g}^k \cdot \boldsymbol{G}_L = g^k{}_L. \tag{1.464}$$

In a similar way, one arrives at

$$X^K{}_{,k} = (\delta^l{}_k - \tilde{e}^l{}_k - \tilde{r}^l{}_k)g^K_{\ l}. \tag{1.465}$$

For small strains and rotations, Equation (1.463) and Equation (1.465) reduce to

$$x^k_{,K} \sim g^k{}_K, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0 \tag{1.466}$$

$$X^K{}_{,k} \sim g^K{}_k, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.467}$$

From Equation (1.352) and Equation (1.353), one finds

$$E_{KL} = e_{kl}x^k_{,K}x^l_{,L} \tag{1.468}$$

which reduces by the use of Equations (1.458), (1.459), (1.466) and (1.467) to

$$\tilde{E}_{KL} \sim \tilde{e}_{kl}g^k{}_K g^l{}_L, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.469}$$

On the basis of Equation (1.457), a similar relationship applies to the infinitesimal rotation

$$\tilde{R}_{KL} \sim \tilde{r}_{kl}g^k{}_K g^l{}_L, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.470}$$

Furthermore, $J = \det x^k_{,K}$. Substituting Equation (1.463) and linearizing yields

$$\begin{aligned} J \sim \tfrac{1}{3!}e^{KLM}e_{klm}[&g^k{}_K g^l{}_L g^m{}_M + g^k{}_K g^l{}_L(\tilde{E}^R{}_M + \tilde{R}^R{}_M)g^m{}_R \\ &+ g^k{}_K g^m{}_M(\tilde{E}^Q{}_L + \tilde{R}^Q{}_L)g^l{}_Q + g^l{}_L g^m{}_M(\tilde{E}^P{}_K + \tilde{R}^P{}_K)g^m{}_P], \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0 \end{aligned} \tag{1.471}$$

where e^{KLM} and e_{klm} are alternating symbols. This is equivalent to

$$J \sim 1 + \tilde{E}^K{}_K \sim 1 + \tilde{e}^k{}_k, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.472}$$

Substituting Equation (1.463) and Equation (1.472) into the relationship between the Cauchy stress and the second Piola–Kirchhoff stress leads to

$$\sigma^{kl} = J^{-1}S^{KL}x^k_{,K}x^l_{,L} \tag{1.473}$$

and linearizing yields

$$\sigma^{kl} \sim S^{KL}[g^k{}_K g^l{}_L(1 - \tilde{e}^m{}_m) + (\tilde{e}^k{}_m + \tilde{r}^k{}_m) g^m{}_K g^l{}_L \\ + (\tilde{e}^l{}_m + \tilde{r}^l{}_m) g^k{}_K g^m{}_L], \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.474}$$

The inverse of Equation (1.474) amounts to

$$S^{KL} \sim \sigma^{kl}[g^K{}_k g^L{}_l(1 + \tilde{e}^m{}_m) - g^L{}_l g^K{}_m(\tilde{e}^m{}_k + \tilde{r}^m{}_k) \\ - g^K{}_k g^L{}_m(\tilde{e}^m{}_l + \tilde{r}^m{}_l)], \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.475}$$

Substituting the above relations into Equation (1.420) yields a linearized expression for the stress:

$$\sigma^{kl} \sim \gamma^{kl}(1 - \tilde{e}^m{}_m) + \gamma^{ml}(\tilde{e}^k{}_m + \tilde{r}^k{}_m) + \gamma^{km}(\tilde{e}^l{}_m + \tilde{r}^l{}_m) \\ - \beta^{kl} T + \sigma^{klmn} \tilde{e}_{mn}, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0 \tag{1.476}$$

where

$$\gamma^{kl} = \gamma^{KL} g^k{}_K g^l{}_L \tag{1.477}$$

$$\beta^{kl} = \beta^{KL} g^k{}_K g^l{}_L \tag{1.478}$$

$$\sigma^{klmn} = \Sigma^{KLMN} g^k{}_K g^l{}_L g^m{}_M g^n{}_N. \tag{1.479}$$

In a similar way, by combining Equation (1.274) and Equation (1.410) one arrives at

$$q^k = -J^{-1} \kappa^{KL} \theta_{,l} x^k_{,K} x^l_{,L}. \tag{1.480}$$

Linearizing yields

$$q^k \sim -\kappa^{kl} \theta_{,l}, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0 \tag{1.481}$$

where

$$\kappa^{kl} = \kappa^{KL} g^k{}_K g^l{}_L. \tag{1.482}$$

Analogous considerations lead to

$$\rho_0 \varepsilon \sim \rho_0 \psi_0 + \rho_0 \eta_0 \theta_{\text{ref}} + \rho_0 c(\theta) T + [\gamma^{kl} + \beta^{kl} \theta_{\text{ref}}] \tilde{e}_{kl} \\ + \frac{1}{2} \sigma^{klmn} \tilde{e}_{kl} \tilde{e}_{mn}, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0 \tag{1.483}$$

$$\eta = \eta_0 + \frac{cT}{\theta_{\text{ref}}} + \frac{\beta^{kl}}{\rho_0} \tilde{e}_{kl}, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0 \tag{1.484}$$

$$\Sigma = \rho_0 \psi_0 - \rho_0 \eta_0 T - \frac{\rho_0 c}{2\theta_{\text{ref}}} T^2 + [\gamma^{kl} - \beta^{kl} T] \tilde{e}_{kl} \\ + \frac{1}{2} \sigma^{klmn} \tilde{e}_{kl} \tilde{e}_{mn}, \quad \|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0. \tag{1.485}$$

The balance equations now read

$$\sigma^{kl}_{\ \ ;k} + \rho(f^l - \dot{v}^l) = 0 \tag{1.486}$$

$$\rho_0 c\dot{T} + \beta^{kl}\theta_{\mathrm{ref}}\dot{\tilde{e}}_{kl} - (\kappa^{kl}T_{,l})_{;k} - \rho_0 h = 0 \tag{1.487}$$

$$\kappa^{kl}T_{,k}T_{,l} \geq 0. \tag{1.488}$$

The derivation for isotropic materials runs along the same lines and yields for $\|\tilde{\boldsymbol{E}}\|, \|\tilde{\boldsymbol{R}}\| \to 0$

$$\sigma^{kl} \sim \gamma(1 - \tilde{e}^m_{\ m})g^{kl} - \beta T g^{kl} + \lambda\tilde{e}^m_{\ m}g^{kl} + 2(\mu+\gamma)\tilde{e}^{kl} \tag{1.489}$$

$$q^k \sim -\kappa T_{,l}g^{kl} \tag{1.490}$$

$$\rho_0\varepsilon \sim \rho_0\psi_0 + \rho_0\eta_0\theta_{\mathrm{ref}} + \rho_0 cT + [\gamma + \beta\theta_{\mathrm{ref}}]\tilde{e}^m_{\ m} + \frac{1}{2}(\lambda+2\mu)I^2_{1\tilde{e}} - 2\mu I_{2\tilde{e}} \tag{1.491}$$

$$\eta = \eta_0 + \frac{cT}{\theta_{\mathrm{ref}}} + \frac{\beta}{\rho_0}I_{1\tilde{e}} \tag{1.492}$$

$$\Sigma = \rho_0\psi_0 - \rho_0\eta_0 T - \frac{\rho_0 c}{2\theta_{\mathrm{ref}}}T^2 + [\gamma - \beta T]I_{1\tilde{e}} + \frac{1}{2}(\lambda+2\mu)I^2_{1\tilde{e}} - 2\mu I_{2\tilde{e}} \tag{1.493}$$

$$\sigma^{kl}_{\ \ ;k} + \rho(f^l - \dot{v}^l) = 0 \tag{1.494}$$

$$\rho_0 c\dot{T} + \beta\theta_{\mathrm{ref}}\dot{I}_{1\tilde{e}} - (\kappa T_{,l}g^{kl})_{;k} - \rho_0 h = 0 \tag{1.495}$$

$$\kappa T_{,k}T_{,l}g^{kl} \geq 0. \tag{1.496}$$

For materials without residual stress and $T = 0$ Equation (1.489) reduces to

$$\sigma^{kl} \sim \lambda\tilde{e}^m_{\ m}g^{kl} + 2\mu\tilde{e}^{kl}. \tag{1.497}$$

Hence,

$$\sigma^k_{\ k} \sim (3\lambda + 2\mu)\tilde{e}^m_{\ m}. \tag{1.498}$$

For a uniform pressure p we have

$$\sigma^k_{\ k} = 3p \tag{1.499}$$

and (see Equation (1.472)),

$$\tilde{e}^m_{\ m} \sim J - 1 \sim \frac{\mathrm{d}v - \mathrm{d}V}{\mathrm{d}V}, \tag{1.500}$$

which is the volume change. Hence,

$$p = (\lambda + \frac{2}{3}\mu)\frac{\mathrm{d}v - \mathrm{d}V}{\mathrm{d}V} \tag{1.501}$$

from which Equation (1.454) follows.

Summarizing, in the small deformation theory, the strain tensors $\boldsymbol{E}$ and $\boldsymbol{e}$ are replaced by their infinitesimal counterparts $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{e}}$. This is only justified for small strains together with small rotations. Therefore, it is better to use the expression *small deformation theory* rather than infinitesimal strain theory. Using the infinitesimal strains and rotations, the constitutive equations and balance laws can be simplified. Notice that the term “infinitesimal” does not apply to other quantities such as stresses. Equations (1.474) and (1.475) show that also in the linear strain theory the second Piola–Kirchhoff and Cauchy stress both exist and are generally different. The derived equations are valid in the spatial frame of reference.

1.14.5 Isotropic elastic materials

In this section, we start again from Equation (1.399) to Equation (1.402) and assume that Σ is isotropic in $\boldsymbol{C}$ but that the resulting stress $\boldsymbol{S}$ is not necessarily linear in $\boldsymbol{E}$. This covers the large family of so-called isotropic hyperelastic models such as neo–Hooke, Mooney–Rivlin, Ogden and many others, used for materials such as rubber and hyperfoam. Because of the isotropy, Σ can only be a function of the invariants of $\boldsymbol{C}$. These will be denoted in the present context by I_1, I_2 and I_3 (dropping the index C for convenience). Accordingly,

$$\Sigma = \Sigma(I_1, I_2, I_3, \theta, \boldsymbol{X}) \tag{1.502}$$

where

$$I_1 = \operatorname{tr}(\boldsymbol{C}) \tag{1.503}$$

$$I_2 = \tfrac{1}{2}[I_1^2 - \operatorname{tr}(\boldsymbol{C}^2)] \tag{1.504}$$

$$I_3 = \det \boldsymbol{C}. \tag{1.505}$$

Consequently, Equation (1.399),

$$\begin{aligned} \boldsymbol{S} = 2\Bigg[\frac{\partial \Sigma}{\partial I_1}(I_1, I_2, I_3, \theta, \boldsymbol{X})\frac{\partial I_1}{\partial \boldsymbol{C}} + \frac{\partial \Sigma}{\partial I_2}(I_1, I_2, I_3, \theta, \boldsymbol{X})\frac{\partial I_2}{\partial \boldsymbol{C}} \\ + \frac{\partial \Sigma}{\partial I_3}(I_1, I_2, I_3, \theta, \boldsymbol{X})\frac{\partial I_3}{\partial \boldsymbol{C}}\Bigg]. \end{aligned} \tag{1.506}$$

Since

$$\frac{\partial I_1}{\partial C_{KL}} = \frac{\partial C_{MN} G^{MN}}{\partial C_{KL}} = G^{KL} \tag{1.507}$$

$$\begin{aligned} \frac{\partial I_2}{\partial C_{KL}} &= \frac{1}{2}\frac{\partial}{\partial C_{KL}}\left[I_1^2 - C_{PQ} C_{MN} G^{PN} G^{QM}\right] \\ &= \frac{1}{2}\left[2 I_1 G^{KL} - C_{MN} G^{KN} G^{LM} - C_{PQ} C^{PL} C^{QK}\right] \\ &= I_1 G^{KL} - C_{MN} G^{KN} G^{LM} \end{aligned} \tag{1.508}$$

$$\frac{\partial I_3}{\partial C_{KL}} = \operatorname{cofactor}(C_{KL}) = \operatorname{cofactor}(C_{LK}) = I_3 (C^{-1})^{KL} \tag{1.509}$$

we obtain,

$$\boldsymbol{S} = 2\left[\frac{\partial \Sigma}{\partial I_1}(I_1, I_2, I_3, \theta, \boldsymbol{X})\boldsymbol{G}^\sharp + \frac{\partial \Sigma}{\partial I_2}(I_1, I_2, I_3, \theta, \boldsymbol{X})(I_1\boldsymbol{G}^\sharp - \boldsymbol{C}^\sharp) + \frac{\partial \Sigma}{\partial I_3}(I_1, I_2, I_3, \theta, \boldsymbol{X})I_3\boldsymbol{C}^{-1}\right]. \tag{1.510}$$

Here, the θ-dependence is not specified yet. Whether the function $\Sigma(I_1, I_2, I_3, \theta, \boldsymbol{X})$ has to satisfy specific requirements to make sense physically will be discussed in Chapter 4 on hyperelastic materials. Since $\boldsymbol{C}^\sharp$ and $\boldsymbol{C}^{-1}$ have the same eigenvectors and the eigenvectors of $\boldsymbol{C}^\sharp$ are not modified by adding or subtracting a multiple of $\boldsymbol{G}^\sharp$, Equation (1.510) shows that $\boldsymbol{S}$ has the same eigenvectors as $\boldsymbol{C}^\sharp$. Consequently, for an isotropic elastic material the principal second Piola–Kirchhoff stress directions coincide with the principal stretch directions.

1.15 Fluids

Solids and fluids are two major classes of materials. Fluids include both liquids and gases. There are several ways in which a fluid can be described. Assume that there is no gravity. Then, the stress in a liquid at rest is zero. If you stir the liquid and wait until there is no motion the stress will again be zero. If you take a container filled with gas at a given pressure, stir the gas without increasing the external pressure and wait till there is no motion the stress reduces to the hydrostatic pressure before stirring. Consequently, the deformation of liquid materials does not induce stress as long as the liquid is at rest and the density is unchanged. Accordingly, the deformation gradient for quasistatic deformations leaving the density unchanged reduces to the shift operator (Eringen 1980):

$$x^k_{,K} = g^k{}_K. \tag{1.511}$$

In a similar way, one arrives at the following simplifications:

$$C_{KL} = x^k_{,K} x^l_{,L} g_{kl} = G_{KL} \tag{1.512}$$

$$\dot{C}_{KL} = 2d_{kl} x^k_{,K} x^l_{,L} = 2d_{kl} g^k{}_K g^l{}_L \tag{1.513}$$

$$\theta_{,K} = \theta_{,k} g^k{}_K. \tag{1.514}$$

Just as for elastic materials, we start from the material formulation of mechanical grade 1 and thermal grade 1, but now we keep the first time derivatives of the mechanical quantities as well:

$$\boldsymbol{S}(\boldsymbol{X}, t) = \boldsymbol{S}(\boldsymbol{C}, \dot{\boldsymbol{C}}, \rho^{-1}, \dot{\rho}, \theta, \nabla_0\theta, \boldsymbol{X}). \tag{1.515}$$

Now, Equation (1.269) yields

$$\sigma^{kl} = J^{-1} x^k_{,K} S^{KL} x^l_{,L} = \frac{\rho}{\rho_0} S^{KL} g^k{}_K g^l{}_L. \tag{1.516}$$

Since (see Equation (1.178))

$$\dot{\rho} = \dot{\overline{\left(\frac{\rho_0}{J}\right)}} = -\frac{\rho_0}{J^2}\dot{J} = -\rho d^k_{\ k} \tag{1.517}$$

and

$$\frac{\partial}{\partial X^K} = \frac{\partial}{\partial x^k} g^k{}_K \tag{1.518}$$

the Cauchy stress takes the form

$$\boldsymbol{\sigma}(\boldsymbol{X}, t) = \boldsymbol{\sigma}(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta, \boldsymbol{X}). \tag{1.519}$$

Since any configuration leaving the density unchanged is undeformed, $\boldsymbol{X}$ can be replaced by $\boldsymbol{x}$:

$$\boldsymbol{\sigma}(\boldsymbol{x}, t) = \boldsymbol{\sigma}(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta, \boldsymbol{x}). \tag{1.520}$$

The principle of objectivity requires that Equation (1.520) does not change its form after applying an arbitrary time-dependent translation, for example, $\boldsymbol{x}(t)$:

$$\boldsymbol{\sigma}(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta, \boldsymbol{x}) = \boldsymbol{\sigma}(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta, \boldsymbol{0}) \tag{1.521}$$

and the explicit dependence on $\boldsymbol{x}$ drops out:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta) \tag{1.522}$$

and similar expressions for $\boldsymbol{q}$, ε and η. Just as in the derivation of the constitutive laws for elastic materials the entropy inequality plays a major role. The spatial formulation of Equation (1.389) reads

$$-\frac{\rho}{\theta}(\dot{\psi} + \dot{\theta}\eta) + \frac{1}{\theta}\boldsymbol{d} : \boldsymbol{\sigma} - \frac{1}{\theta^2}\boldsymbol{q} \cdot \nabla\theta \geq 0 \tag{1.523}$$

where $\psi = \varepsilon - \theta\eta$, Equation (1.387), and

$$\psi = \psi(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta) \tag{1.524}$$

because of similar dependencies of ε and η. The time derivative of ψ reads

$$\dot{\psi} = \frac{\partial\psi}{\partial\boldsymbol{d}} : \dot{\boldsymbol{d}} + \frac{\partial\psi}{\partial\rho^{-1}} \cdot \dot{\overline{\rho^{-1}}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial\nabla\theta} \cdot \dot{\overline{\nabla\theta}}. \tag{1.525}$$

Substituting Equation (1.525) into Equation (1.523) yields

$$\begin{aligned} & -\frac{\rho}{\theta}\frac{\partial\psi}{\partial\boldsymbol{d}} : \dot{\boldsymbol{d}} - \frac{\rho}{\theta}\frac{\partial\psi}{\partial\rho^{-1}}\dot{\overline{\rho^{-1}}} - \frac{\rho}{\theta}\left(\frac{\partial\psi}{\partial\theta} + \eta\right)\dot{\theta} \\ & \qquad - \frac{\rho}{\theta}\frac{\partial\psi}{\partial\nabla\theta} \cdot \dot{\overline{\nabla\theta}} + \frac{1}{\theta}\boldsymbol{d} : \boldsymbol{\sigma} - \frac{1}{\theta^2}\boldsymbol{q} \cdot \nabla\theta \geq 0. \end{aligned} \tag{1.526}$$

Since (see Equation (1.517))

$$\dot{\overline{\rho^{-1}}} = \frac{1}{\rho}\boldsymbol{d} : \boldsymbol{g} \tag{1.527}$$

this is equivalent to

$$-\frac{\rho}{\theta}\frac{\partial \psi}{\partial \boldsymbol{d}} : \dot{\boldsymbol{d}} + \frac{1}{\theta}\boldsymbol{d} : \left(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \rho^{-1}}\boldsymbol{g}\right) - \frac{\rho}{\theta}\left(\frac{\partial \psi}{\partial \theta} + \eta\right)\dot{\theta} - \frac{\rho}{\theta}\frac{\partial \psi}{\partial \nabla \theta} \cdot \dot{\overline{\nabla \theta}} - \frac{1}{\theta^2}\boldsymbol{q} \cdot \nabla \theta \geq 0. \tag{1.528}$$

Since this equation is linear in the time derivatives, it can only be satisfied if the corresponding coefficients reduce to zero:

$$\frac{\partial \psi}{\partial \boldsymbol{d}} = 0 \tag{1.529}$$

$$\eta = -\frac{\partial \psi}{\partial \theta} \tag{1.530}$$

$$\frac{\partial \psi}{\partial \nabla \theta} = 0. \tag{1.531}$$

Hence,

$$\psi = \psi(\rho^{-1}, \theta) \tag{1.532}$$

and Equation (1.528) reduces to

$$\frac{1}{\theta}\boldsymbol{d} : \left(\boldsymbol{\sigma} - \frac{\partial \psi}{\partial \rho^{-1}}\boldsymbol{g}\right) - \frac{1}{\theta^2}\boldsymbol{q} \cdot \nabla \theta \geq 0. \tag{1.533}$$

Defining the pressure p by

$$p = -\frac{\partial \psi}{\partial \rho^{-1}} \tag{1.534}$$

and the dissipative stress by

$$\boldsymbol{t} := \boldsymbol{\sigma} + p\boldsymbol{g} \tag{1.535}$$

we finally arrive at the following equations:

$$p = -\frac{\partial \psi}{\partial \rho^{-1}}(\rho^{-1}, \theta) \tag{1.536}$$

$$\eta = -\frac{\partial \psi}{\partial \theta}(\rho^{-1}, \theta) \tag{1.537}$$

$$\boldsymbol{t} = \boldsymbol{t}(\boldsymbol{d}, \rho^{-1}, \theta, \nabla \theta) \tag{1.538}$$

$$\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{d}, \rho^{-1}, \theta, \nabla \theta) \tag{1.539}$$

$$\varepsilon = \psi(\rho^{-1}, \theta) - \theta\frac{\partial \psi}{\partial \theta}(\rho^{-1}, \theta) \tag{1.540}$$

$$\boldsymbol{\sigma} = -p\boldsymbol{g} + \boldsymbol{t} \tag{1.541}$$

subject to

$$\frac{1}{\theta}\boldsymbol{d} : \boldsymbol{t} - \frac{1}{\theta^2}\boldsymbol{q} \cdot \nabla\theta \geq 0. \tag{1.542}$$

Equation (1.542) implies that $\boldsymbol{t}$ and $\boldsymbol{q}$ must be at least linear in $\boldsymbol{d}$ and $\nabla\theta$ respectively. Equation (1.538) and Equation (1.539) can be replaced by

$$\boldsymbol{t} = \boldsymbol{t}_L(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta) : \boldsymbol{d} \tag{1.543}$$

$$\boldsymbol{q} = -\boldsymbol{\kappa}_L(\boldsymbol{d}, \rho^{-1}, \theta, \nabla\theta) \cdot \nabla\theta \tag{1.544}$$

where $\boldsymbol{t}_L$ is a fourth-order tensor, $\boldsymbol{\kappa}_L$ is a second-order tensor. Notice that the dissipative stress cannot be derived from a potential function, only the hydrostatic part p can. This is a major difference compared to elastic materials. Equation (1.542) is the fluid equivalent of Equation (1.396) for elastic materials.

Because of the principle of objectivity, Equation (1.543) can be further reduced to

$$\boldsymbol{t} = \alpha_0\boldsymbol{g}^\sharp + \alpha_1\boldsymbol{d}^\sharp + \alpha_2(\boldsymbol{d}^2)^\sharp \tag{1.545}$$

where

$$\alpha_K(\rho^{-1}, \theta, \nabla\theta, I_{1d}, I_{2d}, I_{3d}) \tag{1.546}$$

Linearization yields

$$\alpha_0 = \lambda_v(\rho^{-1}, \theta, \nabla\theta)I_{1d} \tag{1.547}$$

$$\alpha_1 = 2\mu_v(\rho^{-1}, \theta, \nabla\theta) \tag{1.548}$$

$$\alpha_2 = 0 \tag{1.549}$$

and one arrives at the well-known stress expressions for linear Stokesian fluids:

$$\boldsymbol{\sigma} = (-p + \lambda_v\boldsymbol{g} : \boldsymbol{d})\boldsymbol{g} + 2\mu_v\boldsymbol{d}. \tag{1.550}$$

For details, the reader is referred to (Eringen 1980).

The energy equation, Equation (1.355), reads in spatial coordinates:

$$\rho\dot{\varepsilon} = \boldsymbol{d} : \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{q} + \rho h. \tag{1.551}$$

Substitution of Equation (1.540) yields

$$\rho\left[-\left(p + \theta\frac{\partial^2\psi}{\partial\rho^{-1}\partial\theta}\right)\dot{\overline{\rho^{-1}}} - \theta\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta}\right] = \boldsymbol{d} : \boldsymbol{\sigma} - \nabla \cdot \boldsymbol{q} + \rho h, \tag{1.552}$$

which reads by the use of Equation (1.527) and Equation (1.534):

$$\rho\theta\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta} + \theta\frac{\partial^2\psi}{\partial\rho^{-1}\partial\theta}\boldsymbol{d} : \boldsymbol{g} + \boldsymbol{d} : \boldsymbol{t} - \nabla \cdot \boldsymbol{q} + \rho h = 0. \tag{1.553}$$

For most gases, $\boldsymbol{t} = 0$ is assumed (no stress dissipation) and Equation (1.553) reduces to

$$\rho\theta\frac{\partial^2\psi}{\partial\theta^2}\dot{\theta} + \theta\frac{\partial^2\psi}{\partial\rho^{-1}\partial\theta}\boldsymbol{d} : \boldsymbol{g} - \nabla \cdot \boldsymbol{q} + \rho h = 0. \tag{1.554}$$