5

Specialized Methods

5.1 INTRODUCTION

The finite strip method, pioneered in 1968 by Y. K. Cheung (1968a,b), is an efficient tool for analyzing structures with regular geometric platform and simple boundary conditions. Basically, the finite strip method reduces a two-dimensional problem to a one-dimensional problem. In some cases computational savings by a factor of 10 or more are possible compared to the finite element method (Cheung and Tham, 1998).

Originally, the finite strip method was designed for rectangular plate problems [similar to Levy's solution; Timoshenko and Woinowsky-Krieger (1971)]. Later, the finite strip method was extended to treat curved plates (Cheung, 1969b), skewed (quadrilateral) plates, folded plates, and box girders. Formulated as an eigenvalue problem, the finite strip method can be applied to vibration and stability problems of plates and shells with relative ease. Finite prism and finite layer methods were also introduced by Cheung and Tham (1998). These methods reduce three-dimensional problems to two- and one-dimensional problems, respectively, by choosing an appropriate choice of displacement functions. In addition, composite structures, such as sandwich panels with cold-formed facings, can be analyzed efficiently by coupling the finite strip method with the finite prism or finite layer methods (Cheung et al., 1982b; Chong, 1986; Chong et al., 1982a,b; Tham et al., 1982).

The finite strip method has been modified through the use of spline functions to analyze plates of arbitrary shape (Cheung et al., 1982a;

Chong and Chen, 1986; Li et al., 1986; Tham et al., 1986; Yang and Chong, 1982, 1984). Complicated boundary conditions can be accommodated. In general, finite strip methods based on spline functions are more involved than the finite strip method based on trigonometric series. However, they are still more efficient than finite element methods, and they require less input and less computational effort.

5.2 FINITE STRIP METHOD

The finite strip method (FSM), first proposed by Y. K. Cheung (1968a,b), is approaching a state of maturity as a structural analysis technique (Puckett et al., 1987; and Wiseman et al., 1987). Two comprehensive books on the method and its applications are available (Cheung and Tham, 1998; Loo and Cusens, 1978), as are papers on advances in the field (Cheung, 1981; and others). The paper by Wiseman et al., (1987), summarizes developments in the methods of finite strips, finite layers, and finite prisms, and includes a review of 114 references.

To examine the finite strip method, consider a rectangular plate with x and y axes in the plane of the plate and axis z in the thickness direction (Fig. 5.1). Let the corresponding displacement components be denoted by (u, v, w). Then similar to Levy's solution (Timoshenko and Woinowsky-Krieger, 1971), for a typical strip (Fig. 5.1), the w displacement component is (Cheung and Tham, 1998).

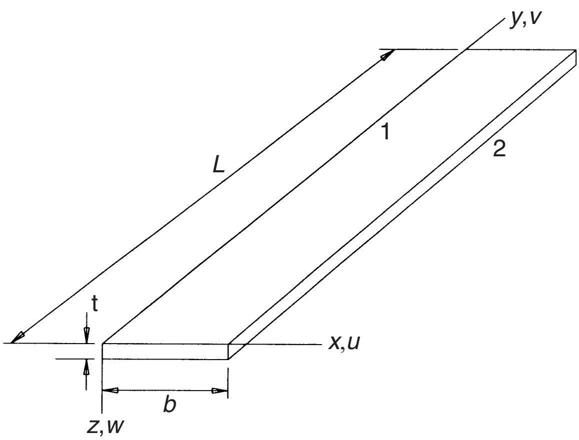

Figure 5.1 Rectangular bending strips.

$$w(x, y) = \sum_{m-1}^{r} f_m(x)Y_m(y) \tag{5.1}$$

in which the functions $f_m(x)$ are polynomials and the functions $Y_m(y)$ are trigonometric terms that satisfy the end conditions in the y direction. The functions Y_m can be taken as basic functions (mode shapes) of the beam vibration equation

$$\frac{d^4Y}{dx^4} - \frac{\mu^4}{a^4} Y = 0 \tag{5.2}$$

where a is the beam (strip) length and μ is a parameter related to frequency, material, and geometric properties.

The general solution of Eq. (5.2) is

$$Y(y) = C_1 \sin \frac{\mu y}{a} + C_2 \cos \frac{\mu y}{a} + C_3 \sinh \frac{\mu y}{a} + C_4 \cosh \frac{\mu y}{a} \tag{5.3}$$

Four boundary conditions are needed to determine the coefficients C_1 to C_4. For example, for both ends simply supported,

$$Y(0) = Y''(0) = Y(a) = Y''(a) = 0 \tag{5.4}$$

Equations (5.3) and (5.4) yield the mode shape functions

$$Y_m(y) = \sin \frac{\mu_m y}{a} \qquad m = 1, 2, 3, \ldots, r \tag{5.5}$$

where $\mu_m = m\pi$; $m = 1, 2, 3, \ldots, r$.

Since the functions Y_m are mode shapes, they are orthogonal (Meirovitch, 1986); that is, they satisfy the relations

$$\int_0^a Y_m Y_n \, dy = 0 \qquad \text{for } m \neq n \tag{5.6}$$

Also, it can be shown that (Cheung and Tham, 1998),

$$\int_0^a Y''_m Y''_n \, dy = 0 \qquad \text{for } m \neq n \tag{5.7}$$

The orthogonal properties of Y_m result in structural matrices with very narrow bandwidths, thus minimizing computational storage and computational time.

Similarly to the finite element method, the functions $f_m(x)$ in Eq. (5.1) can be expressed as

$$f_m(x) = [[C_1] \quad [C_2]] \begin{Bmatrix} \{\delta_1\} \\ \{\delta_2\} \end{Bmatrix}_m \tag{5.8}$$

where subscripts 1 and 2 denote sides 1 and 2 of the plate (strip), respectively; $[C_1]$ and $[C_2]$ are interpolating functions, equivalent to shape functions in one-dimensional finite elements; and $\{\delta_1\}$ and $\{\delta_2\}$ are nodal parameters.

Let b be the width of a strip in the plate and let $\bar{x} = x/b$. Taking the functions $f_m(x)$ as linear functions of x, and considering nodal displacements only, we have

$$\delta_1 = w_1, \qquad \delta_2 = w_2 \qquad C_1 = 1 - \bar{x} \qquad C_2 = \bar{x}$$

Hence, by Eq. (5.8), we have

$$f_m(x) = [(1 - \bar{x}) \quad \bar{x}] \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = (1 - \bar{x})w_1 + \bar{x}w_2 \tag{5.9}$$

Equation (5.9) is equivalent to a one-dimensional linear finite element (Chapter 4), which employs nodal displacements only.

For higher-order functions, with nodal displacements w_i and first derivatives (nodal slopes) $\theta_i = w_i'$, we have

$$\{\delta_1\} = \begin{Bmatrix} w_1 \\ \theta_1 \end{Bmatrix} \qquad \{\delta_2\} = \begin{Bmatrix} w_2 \\ \theta_2 \end{Bmatrix}$$

$$[C_1] = [(1 - 3\bar{x}^2 + 2\bar{x}^3) \quad x(1 - 2\bar{x} + \bar{x}^2)] \tag{5.10}$$

$$[C_2] = [(3\bar{x}^2 - 2\bar{x}^3) \qquad x(\bar{x}^2 - \bar{x})]$$

By Eqs. (5.8) and (5.10), we obtain

$$f_m(x) = [(1 - 3\bar{x}^2 + 2\bar{x}^3) \quad x(1 - 2\bar{x} + \bar{x}^2) \quad (3\bar{x}^2 - 2\bar{x}^3) \quad x(\bar{x}^2 - \bar{x})] \times \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} \tag{5.11}$$

Equation (5.11) is equivalent to a one-dimensional beam element, which employs both nodal displacements and nodal slopes. Other higher-order functions can be derived in a similar manner (Cheung and Tham, 1998).

5.3 FORMULATION OF THE FINITE STRIP METHOD

In general [see Eqs. (5.1) and (5.8)], the displacement function may be written as

$$w = \sum_{m=1}^{r} Y_m \sum_{k=1}^{s} [C_k]\{\delta_k\}_m \tag{5.12}$$

in which r is the number of mode shape functions [Eq. (5.5)] and s is the number of nodal line parameters.

Let

$$[N_k]_m = Y_m[C_k] \tag{5.13}$$

Then, by Eq. (5.12),

$$w = \sum_{m=1}^{r} \sum_{k=1}^{s} [N_k]_m\{\delta_k\}_m = [N]\{\delta\} \tag{5.14}$$

where $[N]$ denotes the shape functions and $\{\delta\}$ denotes the nodal parameters.

The formulation of the finite strip method is similar to that of the finite element method (Section 4.2). For example, for a strip subjected to bending, the strain matrix (vector), $\{\varepsilon\}$, is given by

$$\{\varepsilon\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{Bmatrix} -\partial^2 w/\partial x^2 \\ -\partial^2 w/\partial y^2 \\ 2\partial^2 w/\partial x\, \partial y \end{Bmatrix} \tag{5.15}$$

where M_x, M_y, M_{xy} are moments per unit length.

Differentiating w, Eq. (5.14), and substituting into Eq. (5.15), we obtain the result

$$\{\varepsilon\} = [B]\{\delta\} = \sum_{m=1}^{r} [B]_m\{\delta\}_m \tag{5.16}$$

in which

$$[B]_m = \begin{Bmatrix} -\partial^2[N]_m/\partial x^2 \\ -\partial^2[N]_m/\partial y^2 \\ 2\partial^2[N]_m/\partial x\,\partial y \end{Bmatrix} \tag{5.17}$$

Equation (5.16) is similar to Eq. (4.25).

By Hooke's law (Boresi and Chong, 2000) and Eq. (5.16), the stress matrix (vector), $\{\sigma\}$, is

$$\begin{aligned}\{\sigma\} &= [D]\{\varepsilon\} \\ &= [D]\sum_{m=1}^{r} [B]_m\{\delta\}_m\end{aligned} \tag{5.18}$$

where $[D]$ is the elasticity matrix, defined in Section 4.2 [Eqs. (4.10), (4.11), and (4.12)] for isotropic plane strain and plane stress problems.

Minimization of the total potential energy gives (Cheung and Tham, 1998),

$$[K]\{\delta\} = \{F\} \tag{5.19}$$

where

$$[K] = \int_{\text{vol}} [B]^{\text{T}}[D][B]\,dv \tag{5.20}$$

is the stiffness matrix and $\{F\}$ is the load vector.

For a distributed load $\{q\}$, the load vector is

$$\{F\} = \int_A [N]^{\text{T}}\{q\}\,dA \tag{5.21}$$

Expanding Eq. (5.20), we obtain the stiffness matrix in the form

$$[K] = \int_{\text{vol}} [[B]_1^{\text{T}}[B]_2^{\text{T}} \cdots [B]_r^{\text{T}}][D][[B]_1[B]_2 \cdots [B]_r]\, dv$$

$$= \int_{\text{vol}} \begin{bmatrix} [B]_1^{\text{T}}[D][B]_1 & [B]_1^{\text{T}}[D][B_2] \cdots & [B]_1^{\text{T}}[D][B]_r \\ \vdots & \vdots & \vdots \\ [B]_r^{\text{T}}[D][B]_1 & [B]_r^{\text{T}}[D][B]_2 \cdots & [B]_r^{\text{T}}[D][B]_r \end{bmatrix} dv$$

$$= \begin{bmatrix} [k]_{11} & [k]_{12} \cdots & [k]_{1r} \\ \vdots & \vdots & \vdots \\ [k]_{r1} & [k]_{r2} \cdots & [k]_{rr} \end{bmatrix} \tag{5.22}$$

where

$$[k]_{mn} = \int_{\text{vol}} [B]_m^{\text{T}}[D][B]_n\, dv \tag{5.23}$$

For each strip (with s nodal line parameters),

$$[k]_{mn} = \begin{bmatrix} [k_{11}] & [k_{12}] \cdots & [k_{1s}] \\ \vdots & \vdots & \vdots \\ [k_{s1}] & [k_{s2}] \cdots & [k_{ss}] \end{bmatrix}_{mn}$$

In general, the individual elements of matrix $[k]_{mn}$ are given by

$$[k_{ij}]_{mn} = \int_{\text{vol}} [B_i]_m^{\text{T}}[D][B_j]_n\, dv \tag{5.24}$$

Similarly the basic element of the load vector (for distributed loads) is

$$\{F_i\} = \int_A [N_i]_m^{\text{T}}\{q\}\, dA \tag{5.25}$$

For simple functions Eqs. (5.24) and (5.25) can be evaluated in closed form (Cheung and Tham, 1998). Alternatively, they can be integrated numerically using Gaussian quadrature or other numerical integration methods.

5.4 EXAMPLE OF THE FINITE STRIP METHOD

In this section the finite strip method is applied to the analysis of a uniformly loaded plate simply supported on two parallel edges. Each nodal line is free to move in the z direction and rotate about the y axis (Fig. 5.1). Thus Eqs. (5.1), (5.5), and (5.10) are to be used. The matrix $[B]_m$, Eqs. (5.17) and (5.23), is (Cheung and Tham, 1998).

$$[B]_m = \begin{bmatrix} \frac{6}{b^2}(1-2\bar{x})Y_m & \frac{2}{b}(2-3\bar{x})Y_m & \frac{6}{b^2}(-1+2\bar{x})Y_m & \frac{2}{b}(-3\bar{x}+1)Y_m \\ -(1-3\bar{x}^2+2\bar{x}^3)Y''_m & -x(1-2\bar{x}+\bar{x}^2)Y''_m & -(3\bar{x}^2-2\bar{x}^3)Y''_m & -x(\bar{x}^2-\bar{x})Y''_m \\ \frac{2}{b}(-6\bar{x}+6\bar{x}^2)Y'_m & 2(1-4\bar{x}+3\bar{x}^2)Y'_m & \frac{2}{b}(6\bar{x}-6\bar{x}^2)Y'_m & 2(3\bar{x}^2-2\bar{x})Y'_m \end{bmatrix}$$

For orthotopic plates, the elasticity matrix is (Boresi and Chong, 2000; Cheung and Tham, 1998)

$$[D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \tag{5.26}$$

where

$$D_x = \frac{E_x t^3}{12(1-\nu_x\nu_y)} \qquad D_y = \frac{E_x t^3}{12(1-\nu_x\nu_y)} \qquad D_{xy} = \frac{Gt^3}{12}$$

$$D_1 = \frac{\nu_x E_y t^3}{12(1-\nu_x\nu_y)} = \frac{\nu_y E_x t^3}{12(1-\nu_x\nu_y)} \tag{5.27}$$

For isotropic materials,

$$E_x = E_y = E, \quad \nu_x = \nu_y = \nu \quad \text{and} \quad G = \frac{E}{2(1 + \nu)} \tag{5.28}$$

By the orthogonal properties, $[k]_{mn} = 0$ if $m \neq n$. Thus, Eq. (5.19) reduces to

$$\begin{bmatrix} [k]_{11} & & & \\ & [k]_{22} & & \\ & & \ddots & \\ & & & [k]_{rr} \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ \vdots \\ w_{r+1} \\ \theta_{r+1} \end{Bmatrix} = \begin{Bmatrix} \{F_1\} \\ \\ \{F_2\} \\ \vdots \\ \{F_r\} \end{Bmatrix} \tag{5.29}$$

where the matrices $[k]_{mm}$ are 4×4 symmetrical matrices (Cheung and Tham, 1998). The load vector $\{F_m\}$ can be computed by integrating Eq. (5.25). It is a 4×1 matrix given by

$$\{F\} = q \begin{Bmatrix} b/2 \\ b^2/12 \\ b/2 \\ -b^2/12 \end{Bmatrix} \int_0^a \sin \frac{m\pi y}{a} \, dy \tag{5.30}$$

The load matrices $\{F\}_m$ for eccentric and concentric concentrated loads have been derived by Cheung and Tham (1998). With two degrees of freedom (w, θ) along the nodal line, the moments [Eq. (5.15)] are linear functions of x. Thus, to obtain the maximum moment at a known location (e.g., at the center for uniform load and under load point for concentrated load), a nodal line must pass through this location for good accuracy. In a convergence test applied to a uniformly loaded simply supported square plate, Cheung and Tham, 1998) have shown that with four strips (10 equations), greater accuracy was obtained than that by the finite element method with 25 nodes (75 equations). In addition, considerable saving in computational labor is achieved (Yang and Chong, 1984). Higher-order strips have also been applied to plates (Cheung and Tham, 1998). Finite strip formulations for curved and skewed plates have also been derived (Cheung and Tham, 1998). These techniques are applicable to the analysis of curved bridges, stiffened plates, and so on.

5.5 FINITE LAYER METHOD

By selecting functions satisfying the boundary conditions in two directions, the philosophy of the finite strip method can be extended to layered systems. The resulting method is called the *finite layer method* (FLM). The method was first proposed by Cheung and Chakrabarti (1971). The finite layer method is useful for layered materials, rectangular in planform. To illustrate the method, consider Fig. 5.2. Let $\bar{z} = z/c$. Then the lateral displacement component, w, can be expressed as

$$w = \sum_{m=1}^{r} \sum_{n=1}^{t} [(1 - \bar{z})w_{1mn} + \bar{z}w_{2mn}]X_m(x)Y_n(y) \qquad (5.31)$$

in which, w_{1mn} and w_{2mn} are displacement parameters for side 1 (top) and side 2 (bottom), respectively. The displacement component, w, is assumed to vary linearly in the z direction. The functions X_m and Y_n are taken as terms in a trigonometric series, satisfying the boundary conditions. In this manner a three-dimensional problem is reduced to a one-dimensional problem with considerable saving in computer storage and computational time (Cheung, and Tham, 1998; Cheung et al., 1982b).

By linear theory, the (x, y) displacement components (u, v) are linearly related to the derivatives of w; that is,

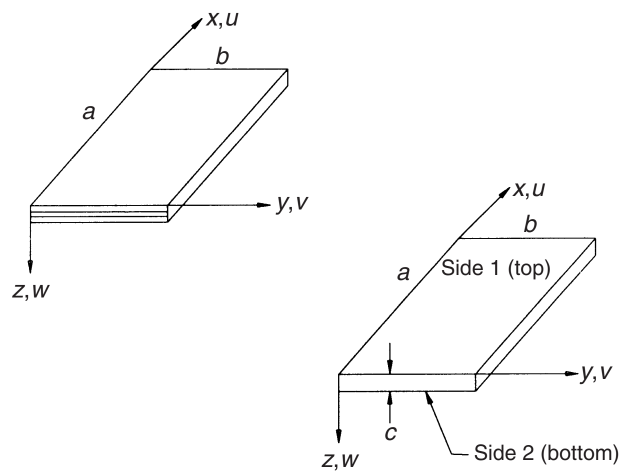

Figure 5.2 Finite layers.

$$u = A \frac{\partial w}{\partial x} \tag{5.32}$$

$$v = B \frac{\partial w}{\partial y} \tag{5.33}$$

Thus, as with Eq. (5.31), we may write

$$\begin{aligned} v &= \sum_{m=1}^{r} \sum_{n=1}^{t} [(1 - \bar{z})v_{1mn} + \bar{z}v_{2mn}]X_m(x)Y'_n(y) \\ u &= \sum_{m=1}^{r} \sum_{n=1}^{t} [(1 - \bar{z})u_{1mn} + \bar{z}u_{2mn}]X'_m(x)Y_n(y) \end{aligned} \tag{5.34}$$

in which u_{1mn}, v_{1mn}, u_{2mn} and v_{2mn} are displacement parameters of sides 1 and 2.

The displacement is

$$\{f\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{m=1}^{r} \sum_{n=1}^{t} [N]_{mn}\{\delta\}_{mn} = [N]\{\delta\} \tag{5.35}$$

where

$$\{\delta\}_{mn} = [u_{1mn}, v_{1mn}, w_{1mn}, u_{2mn}, v_{2mn}, w_{2mn}]^{\mathrm{T}} \tag{5.36}$$

The strain–displacement relationship is (Boresi and Chong, 2000)

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \\ \frac{1}{2}(\partial u/\partial y + \partial v/\partial x) \\ \frac{1}{2}(\partial v/\partial z + \partial w/\partial y) \\ \frac{1}{2}(\partial w/\partial x + \partial u/\partial z) \end{Bmatrix} = \sum_{m=1}^{r} \sum_{n=1}^{t} [B]_{mn}\{\delta\}_{mn} = [B]\{\delta\} \tag{5.37}$$

The stress–strain relationship is

$$\{\sigma\} = [\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \sigma_{xy}\ \sigma_{yz}\ \sigma_{zx}]^{\mathrm{T}} = [D][B]\{\delta\} \tag{5.38}$$

Proceeding as in Section 5.4, we obtain the stiffness matrix (Cheung and Tham, 1998)

$$[S] = \int_0^a \int_0^b \int_0^c [B]^{\mathrm{T}}[D][B]\, dx\, dy\, dz \tag{5.39}$$

For a simply supported rectangular plate, we have the conditions

$$\int_0^a \int_0^b \int_0^c [B]_{mn}^{\mathrm{T}}[D][B]_{rs}\, dx\, dy\, dz = 0 \qquad \text{for } mn \neq rs \tag{5.40}$$

Hence, the off-diagonal terms are zero, and the stiffness matrix reduces to

$$[K] = \int_0^a \int_0^b \int_0^c \begin{bmatrix} [B]_{11}^{\mathrm{T}}[D][B]_{11} & & & & & \\ & [B]_{12}^{\mathrm{T}}[D][B]_{12} & & & & \\ & & \ddots & & & \\ & & & [B]_{1t}^{\mathrm{T}}[D][B]_{1t} & & \\ & & & & [B]_{21}^{\mathrm{T}}[D][B]_{21} & \\ & & & & & \ddots \\ & & & & & & [B]_{rt}^{\mathrm{T}}[D][B]_{rt} \end{bmatrix} dx\, dy\, dz \tag{5.41}$$

The remaining formulation is similar to that of Sections 5.3 and 5.4.

5.6 FINITE PRISM METHOD

Similarly to Eq. (5.1), for prismatic members with rectangular planform, the three-dimensional problem can be reduced to a two-dimensional problem by expressing the displacement function f as (Cheung and Tham, 1998)

$$f = \sum_{m=1}^{r} f_m(x, z) Y_m \tag{5.42}$$

where $f_m(x, z)$ is a function of (x, z) only and Y_m are trigonometric functions of y, which satisfy the boundary conditions in the y direction. In formulating the finite prism method (FPM), including nodal displacements and shape functions as in the finite element method, it is convenient to use nodal coordinates. Therefore, let ξ, η be the local coordinates of an element, ξ_k, η_k the nodal coordinates of the element, δ the displacement of a point in the element, δ_k the nodal displacements, and (ϕ_k, ψ_k) functions associated with a particular coordinate system (such as Cartesian, skew, or curvilinear). Then we can represent the local coordinates in terms of nodal coordinates in the form

$$\xi = \sum_{k=1}^{s} \phi_k \xi_k \tag{5.43}$$

where s refers to the number of nodes of the element.

Similarly, the displacement δ of a point within the element can be expressed in terms of the nodal displacements δ_k as

$$\delta = \sum_{k=1}^{s} \psi_k \delta_k \tag{5.44}$$

In general, $\phi_k \neq \psi_k$; however, if $\phi_k = \psi_k$, the element is termed *isoparametric*. Using Eq. (5.42), for a prismatic member with two ends simply supported, the lateral (out-of-plane) displacement component is

$$w = \sum_{m=1}^{r} w_m(x, z) \sin k_m y \tag{5.45}$$

in which

$$w_m(x, z) = \sum_{k=1}^{s} C_k w_{km} \tag{5.46}$$

and w_{km} are the mth term nodal displacements at the kth node. the C_k are the shape functions for the two-dimensional element (in the ξ–η plane). For an isoparametric six-node (ISW′6) model (Fig. 5.3), the shape functions are given as follows:

Corner nodes:

$$C_k = \tfrac{1}{4}\eta_k \eta(1 + \eta_k \eta)(1 + \xi_k \xi) \tag{5.47}$$

Midside nodes:

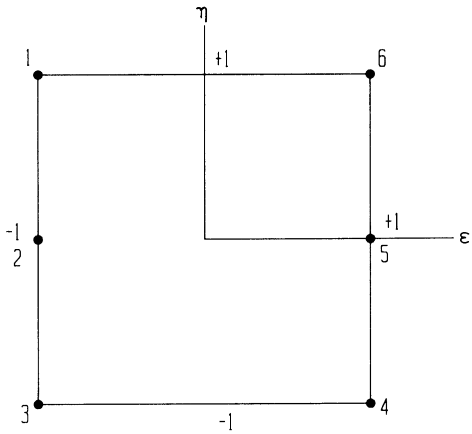

Figure 5.3 ISW'6 model.

$$C_k = \tfrac{1}{2}(1 + \xi_k \xi)(1 - \eta^2) \tag{5.48}$$

Therefore, the lateral (out-of-plane) displacement component for the isoparametric six-node model is

$$w = \sum_{m=1}^{r} \sum_{k=1}^{6} C_k w_{km} \sin k_m y \tag{5.49}$$

The corresponding in-plane displacement components (u, v) are

$$u = \sum_{m=1}^{r} \sum_{k=1}^{6} C_k u_{km} \sin k_m y \tag{5.50}$$

$$v = \sum_{m=1}^{r} \sum_{k=1}^{6} C_k v_{km} \cos k_m y \tag{5.51}$$

The stiffness matrix is developed in a manner similar to the development in Sections 5.3 and 5.4. It is (Chong et al., 1982b)

$$^{\mathrm{P}}K_{ijmn} = \int {}^{\mathrm{P}}B_{im}^{\mathrm{T}} \, {}^{\mathrm{P}}D \, {}^{\mathrm{P}}B_{jn} \, d(\mathrm{vol}) \tag{5.52}$$

where

$$
{}^{P}B_{im} = \begin{bmatrix} \frac{\partial C_i}{\partial x}\sin\frac{m\pi y}{L} & 0 & 0 \\ 0 & -C_i\frac{m\pi}{L}\sin\frac{m\pi y}{L} & 0 \\ 0 & 0 & \frac{\partial C_i}{\partial z}\sin\frac{m\pi y}{L} \\ C_i\frac{m\pi}{L}\cos\frac{m\pi y}{L} & \frac{\partial C_i}{\partial x}\cos\frac{m\pi y}{L} & 0 \\ 0 & \frac{\partial C_i}{\partial x}\cos\frac{m\pi y}{L} & C_i\frac{m\pi}{L}\cos\frac{m\pi y}{L} \\ \frac{\partial C_i}{\partial x}\sin\frac{m\pi y}{L} & 0 & \frac{\partial C_i}{\partial z}\sin\frac{m\pi y}{L} \end{bmatrix} \tag{5.53}
$$

${}^{P}D$ is the elasticity matrix for isotropic materials. In Eq. (5.52), the superscript P indicates the finite prism model.

The geometric transformation from the natural coordinate (x, z) to the local coordinate (ξ, η) can be carried out as in the standard finite element method, and the stiffness matrix can be obtained accordingly (Cheung and Tham, 1998).

5.7 APPLICATIONS AND DEVELOPMENTS OF FSM, FLM, AND FPM

Owing to their narrow bandwidth and reduction in dimensions, the finite strip, finite layer, and finite prism methods (FSM, FLM, and FPM) are especially adaptable for personal computers (Rhodes, 1987) and minicomputers, with significant savings in computational labor. Developments and applications are presented by Wiseman et al. (1987) and Graves-Smith (1987).

One of the first reported applications of the method was to orthotropic right box girder bridge decks (Cheung, 1969a). Cheung suggested that the finite strip method could be used in a composite slab-beam system, where the longitudinal beam stiffness could be derived separately and added to the structure stiffness. This procedure is a refinement to systems in which the action of longitudinal beams is approximated by assuming orthotropic properties for the slab. Cheung further asserted that the beam would have to coincide with a strip nodal line. This condition was verified by comparison with theoretical and model test results for simply supported slab-beam models with varying slab thicknesses (Cheung et al., 1970). A force method for introducing rigid column supports in the strips was also described.

An analytical method using a separation-of-variables procedure and series solution for the resulting differential equations was developed by Harik and his associates (1984, 1985, 1986) for orthotropic sectorial plates and for rectangular plates under various loadings. The plates were divided into strips where point and patch loads were applied, and the accuracy of the solutions was demonstrated for various boundary conditions.

Two refinements aimed at increasing the accuracy of finite strips in the in-plane transverse direction were developed by Loo and Cusens (1970). One refinement ensued curvature compatibility at the nodal lines. Although this requirement limited the analysis to plates with uniform properties across the section, convergence was shown to be faster for appropriate plate systems. The other refinement introduced an auxiliary nodal line within the strips. A drawback of these refinements is that they both increase the size of the stiffness matrix by 50%.

Other formulations for increasing the accuracy or the range of applicability of the finite strip method have been published. Brown and Ghali (1978) used a subparametric finite strip for the analysis of quadrilateral plates. Bucco et al. (1979) proposed a deflection contour method which allows the finite strip method to be used for any plate shape and loading for which the deflection contour is known.

Other applications of the finite strip method have used spline functions for the boundary component of the displacement function. Yang and Chong (1982, 1984) used *X*-spline functions, allowing extension of the finite strip method to plate bending problems with irregular boundaries. The solution was shown to converge correctly for a trapezoidal plate, and techniques for approximating a plate with more irregularly shaped sides and ends were discussed. The buckling of irregular plates has been investigated by Chong and Chen (1986). Extensive development of the finite strip method using the cubic *B*-spline as the boundary component of the displacement function was done by Cheung et al. (1982a). After initial application to flat plates and box girder bridges by Cheung and Fan (1983), the cubic *B*-spline method was extended to skewed plates by Tham et al (1986), to curved slabs by Cheung et al. (1986) and to arbitrarily shaped general plates by Li et al. (1986). Shallow shells were analyzed by Fan and Cheung (1983) using a spline finite strip formulation based on the theory of Vlasov (1961).

The application of the finite strip method to folded plates by Cheung (1969c) extended the method to three-dimensional plate structures. The strips used were simply supported with respect to out-of-plane displacement and had similar in-plane longitudinal displacement functions. These end conditions suggest a real plate system having end diaphragms or supports with infinite in-plane stiffness. This system is an acceptable approximation for many structures, including roof systems resting on walls and box girder bridges with plate diaphragms.

A study of folded plates, continuous over rigid supports, has been presented by Delcourt and Cheung (1978). They represented longitudinal displacement functions as eigenfunctions of a continuous beam with the same span and relative span stiffnesses as the plate structure. For comparison, the longitudinal displacement function for a structure with pinned ends and no internal supports is a sine function.

The finite strip method has become popular for the analysis of slab girder and box girder bridges. Cheung and Chan (1978) used folded plate elements to study 392 theoretical bridge models in order to suggest refinements to American and Canadian design codes. Cusens and Loo (1974) have done extensive work in the application of the finite strip method to the analysis of prestressed concrete box girder bridges, and their book (Loo and Cusens, 1978) contains an extensive treatment of the subject with many examples.

The analysis of cylindrically orthotropic curved slabs using the finite strip method was described by Cheung (1969c). The primary difference between this technique and the finite strip method for right bridge decks is the use of a polar coordinate system for the displacement function. It was noted that a model in this coordinate system could be used to approximate a rectangular slab by specifying a very large radius for the plate. A radial shell element was subsequently developed (Cheung and Cheung, 1971a) which allowed the analysis of curved box girder bridges supported by rigid end diaphragms.

Because of its efficiency and ease of input, the finite strip method has been readily adapted to free vibration and buckling problems. The formulation for frequency analysis was outlined by Cheung et al. (1971). Characteristic functions for strips with varying end conditions have also been developed (Cheung and Cheung, 1971b). Finite strips and prisms were combined by Chong et al. (1982a,b) to study free vibrations and to calculate the effects of different face temperatures on foam-core architectural sandwich panels (Fig. 5.4). The panel facings were modeled by finite strips and the panel core by finite prisms. The eigenvalue problem for free vibrations is characterized by the equation (Chong et al., 1982a)

$$-\overline{\omega}^2\,{}^{S}M_m^{P}\,{}^{S}\overline{\delta}_m^{P} - \overline{\omega}^2\,{}^{S}M_m^{B}\,{}^{S}\overline{\delta}_m^{B} - \overline{\omega}^2\,{}^{P}M_m\,{}^{P}\overline{\delta}_m + {}^{S}K_m^{P}\,{}^{S}\overline{\delta}_m^{P} + {}^{S}K_m^{B}\,{}^{S}\overline{\delta}_m^{B} + {}^{P}K_m\,{}^{P}\overline{\delta}_m = 0 \quad (5.54)$$

where $\overline{\omega}$ is the frequency, ${}^{S}M_m^{P}$ is the in-plane mass matrix of the bending strip, ${}^{S}M_m^{B}$ is the bending mass matrix of the bending strip, ${}^{P}M_m$ is the mass matrix of the finite prism, ${}^{S}K_m^{P}$ is the in-plane stiffness matrix of the bending strip, ${}^{S}K_m^{B}$ is the bending stiffness matrix of the bending strip, ${}^{P}K_m$ is the stiffness matrix of the finite prism, ${}^{S}\overline{\delta}_m^{P}$ and ${}^{S}\overline{\delta}_m^{B}$ are the in-plane and bending interpolation parameters of the bending strip respectively, and ${}^{P}\overline{\delta}_m$ are the interpolating parameters of the finite prism.

Yoshida (1971) was among the first to develop the geometric stiffness matrix for buckling of rectangular plates. Przemieniecki (1972) used the same displacement function as in the finite strip method (Cheung and Tham, 1998) for the analysis of local stability in plates and stiffened panels. Chong and Chen (1986) investigated the buckling of nonrectangular plates by spline finite strips. Cheung et al. (1982b) used the finite layer method to derive buckling loads of sandwich plates. Mahendran and Murray (1986) modified the standard finite strip displacement function to include out-of-phase displacement parameters, allowing shearing loads to be applied to the strips in the buckling analysis.

The finite strip method has been used as the basis for parametric and theoretical studies of column buckling. Both local and overall buckling of H-columns with residual stress under axial load were studied by Yoshida and Maegawa (1978). Buckling of columns with I-sections was

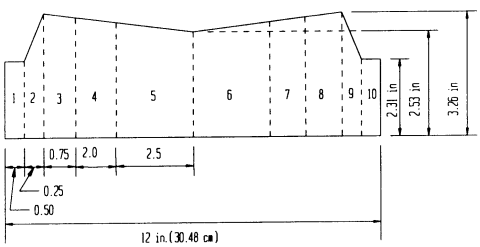

Figure 5.4 Finite strips and prisms in a sandwich panel.

studied by Hancock (1981). The interaction of the two modes of buckling for other prismatic columns was also investigated by Sridharan and Benito (1984) using the finite strip method. Hasegawa and Maeno (1979) determined the effects of stiffeners on cold-formed steel shapes using finite strips. The postbuckling behavior of columns has been the subject of papers by Graves-Smith and Sridharan (1979), Sridharan and Graves-Smith (1981), and Graves-Smith and Gierlinski (1982).

The compound strip method, an enhancement of the flexibility and applicability of the finite strip method, was developed by Puckett and Gutkowski (1986). This formulation added both longitudinal and transverse beams at any location in a strip and included the effect of column supports. A stiffness formulation, which included the added stiffness of all elements associated with a beam, eliminated the repetitive solution passes required for each redundant in a flexibility analysis. The strain energies for flexure and torsion of the beams as well as for axial and flexural deformation of the columns were derived in terms consistent with the strip displacement function, allowing the total strip stiffness to be calculated by adding the element stiffnesses. With the compound strip method, the analysis of continuous beams can be performed using simple displacement functions previously suitable only for single spans. The compound strip method was extended to curved plate systems and was used for dynamic analysis by Puckett and Lang (1986).

Gutkowski et al. (1991) developed a cubic *B*-spline finite strip method for the analysis of thin plates. Spline series with unequal spacing allow local discretization refinement near patch and concentrated loads. Oscillatory convergence (Gibb's phenomenon) is avoided. Spline finite strips have also been used to analyze non-prismatic space structures (Tham, 1990).

Transient responses of noncircular cylindrical shells subject to shock waves were studied by Cheung et al. (1991), using finite strips and a coordinate transformation. Arbitrarily shaped sections were mapped approximately into circular cylindrical shells in which finite strips can be applied for discretization. The finite strip method was used for the free vibration and buckling analysis of plates with abrupt thickness changes and a nonhomogeneous winkler elastic foundation (Cheung et al., 2000).

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