

## LESSON 1

### What Algebra Is

In these lessons, we are going to explore key moments in the development of algebra in different places over the past 3500 years. As we shall see, different people have written about algebra in different ways, depending on the kinds of problems they were solving and the ways in which they manipulated numbers. In order to get a perspective that will enable us to appreciate what all these writings have in common, we devote this first lesson and the one following to some very general considerations. In the present lesson, we explore the nature of algebra itself and the different number systems in which its problems are stated and solved.

#### 1. Numbers in disguise

As human societies grow larger, their administrative complexity grows disproportionately. While a single leader can make all the decisions on where to hunt, where to encamp, how to watch out for enemies, and so on for a small clan in which everyone knows everyone else, large societies, in which people must often deal with strangers, require formal laws to govern behavior. As economies become more complex, it is necessary to regulate commerce, weights, and measures and to plan strategically for defense or conquest. Over time, a group of specialized bureaucrats arises, charged with administering these vital activities.

These bureaucrats universally rely on two forms of mathematics, arithmetic and geometry. To collect taxes on land, to regulate trade and agriculture, to design and construct large public works, it is vital to know the elements of these two subjects. Records show that the people of Egypt and Mesopotamia possessed this knowledge at least 4000 years ago. Undoubtedly, such knowledge was also current in China and India about the same time. However, there is evidence that the Chinese used mechanical methods of calculating, in the form of counting rods, rather than graphical methods, and thus the details of their mathematics have vanished. Whether for that reason or because the first Emperor Ch'in Shih Huang Ti ordered the burning of all books when he unified China in 221 BCE, only a few Chinese texts known to be more than 2000 years old have been preserved.

Although the term *bureaucrat* has an unfortunate connotation that suggests a soulless automaton, mindlessly enforcing rules, the bureaucrats of these early societies were, like all human beings, possessed of an imagination, and they were the first people who were given economic support that

enabled them to indulge their imagination. They must have been encouraged to plan strategically; not only to see that the current year's harvest is properly stored, distributed, and taxed but also to consider the possibilities of external aggression, future bad weather, and the like. If they were asked to design monuments, bridges, roads, and tunnels, such tasks would exercise their imaginations.

Perhaps in the intervals of their administrative work they found time to play games with the mathematical knowledge they possessed, posing problems for one another. This last activity may well explain why the earliest texts contain so many examples of problems for which a practical application is difficult to imagine. Or perhaps the explanation is our own lack of imagination about the kinds of practical problems they actually faced. Whichever is the case, we find arithmetic and geometry combining in many of these early texts to produce what we might call mathematical riddles, or perhaps *numbers in disguise*, with a challenge to unmask the numbers and make them reveal themselves, as in the following fictional anecdote.

**Example 1.1.** The dynasty of Uresh-tun was the wonder of its neighbors because of the prodigiously tall tree that grew just outside the walls of the king's castle. The kings of this dynasty held court under its branches in pleasant weather. No one knew what kind of tree it was; there was none like it for hundreds of miles around. Then, during the reign of the seventh king of the dynasty of Uresh-tun, this marvelous tree was blown over by a storm and fell with a great crash. The king commanded that it be cut into planks for his own use, and this was done. The largest of these planks was perfectly straight and of even thickness throughout and measured 44 meters in length and 75 centimeters in width. What suitable use could the king make of such a treasure? It was too long to fit inside any of his buildings, and he did not wish to leave it outside to rot in the damp weather.

After much thought, he decided on a use for it: It would furnish the frame for a set of portraits of himself and his six illustrious predecessors of the dynasty. He summoned his artisans and ordered them to cut notches at the ends and at three other points in such a way that the four pieces would provide a single frame for seven identical square tiles on which the portraits would be painted.

The artisans recognized that they must cut out three isosceles right triangles at three points on the plank and two others half as large, one at each end. Where should the three interior cuts be made? Obviously, one of them should be exactly in the middle. But where should the other two go? They could see that removing the triangles at the two ends would decrease the perimeter of the inside of the frame by 1.5 meters, and each of the other three cuts would remove another 1.5 meters, so that the rectangular inside of the frame would have a perimeter of 38 meters. The problem was to make that inner rectangle seven times as wide as it was high.

The folk wisdom of Uresh-tun said, "Measure twice before cutting once," and they knew that the king would not forgive any bungling on their part.

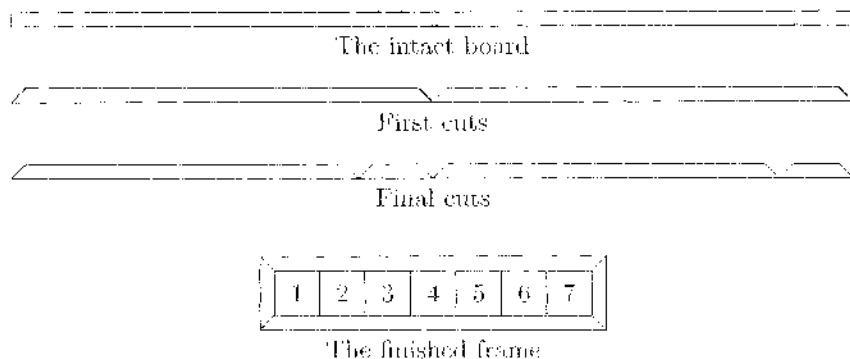


FIGURE 1. Cutting a board to make a picture frame.

They dared not experiment on such a precious piece of wood, and there was no other piece of such length on which they could make practice cuts. They had to get it right the first time. Symmetry showed them that the left and right halves of the plank would have to be cut identically. The problem that remained was to divide a length of 19 m into two parts so that one of the parts was seven times the other.

That is where we leave the artisans. You may enjoy thinking of both experimental and computational ways by which they might solve this problem. Probably you will agree that the computational way is somehow “neater” and more satisfying than trial and error, and much faster, once you see how to do the problem. To visualize it, look at Fig. 1.

Having seen first-hand in histories of mathematics how easily urban legends and folk tales begin, I do not wish to be the source of any new ones. Hence I emphasize again that this example is pure fiction. As far as I know, there has never been any place called Uresh-tun anywhere, much less one that generated the problem just described. However, the pure mathematics problem that corresponds to it was stated by an Egyptian scribe nearly 4000 years ago: *A quantity and its seventh part together equal 19. What is the quantity?*

If you wish to see how the Egyptian scribe solved this problem, look ahead to Lesson 3. However, try to solve it yourself, by both practical and mathematical means. There are several ways to proceed. (See Problem 1.11.)

**1.1. “Classical” and modern algebra.** The carpentry problem just posed leads to a single linear equation in one unknown. As such, it can be solved by pure arithmetic, and so marks the borderline between arithmetic and algebra. You don’t *have* to introduce an equation to solve this problem, although you can if you wish.

What this kind of problem reveals is that numbers do not have to be named explicitly in order to be determined. They are sometimes determined by properties that they have. This way of thinking can apply to any objects, not just numbers. In geometry, lines are often determined by certain properties, such as being tangent to a circle at a given point. The technique of thinking in terms of descriptions, which is the essence of the early algebra we will be describing, was mentioned by the fourth-century geometer Pappus of Alexandria (ca. 290–ca. 350) in Book 7 of his *Collection*. After explaining that analysis proceeds from the object being sought to something that was agreed on (known to be true), he said, “For in analysis we set down the object being sought as something that has been constructed, and then examine what follows from this; then we repeat with that consequence, until by such considerations we arrive at something either already known or some first principle.” He was thinking of geometric objects, but his *analysis* reflects the same kind of thinking used in algebra, where we write down a symbol for the unknown number as if it were already at hand, and then consider the conditions that it must satisfy. In our board-cutting example, the unknown number is characterized as being seven times the difference between 19 and the number itself.

The technique described by Pappus lies at the heart of even the more advanced and subtle thinking involved in the general solution of polynomial equations. Although no general method for finding the roots of a fifth-degree equation was known in the early nineteenth century, nevertheless mathematicians could write down five symbols to represent those roots and reason about the properties they must have. The result was eventually a proof that no finite algebraic formula expressing them exists.

Thus, numbers may appear in disguise, and this way of thinking about them forms the subject that we are going to call *classical algebra*. By that term, we mean the algebra that was practiced in many parts of the world for about 4000 years, from the earliest times to the midnineteenth century. This algebra was confined to the study of polynomial equations, an example of which is the quartic (fourth-degree) equation

$$x^4 - 10x^3 + 3x^2 + 2x - 7 = 0.$$

By the year 1850 the major questions in classical algebra had received answers, and that is the portion of the story of algebra that will be told in this book.

When difficult mathematical problems that have been open for a long time are finally solved, the techniques that were used to solve them generate their own interesting questions and become the foundation of a new subject. In this case that subject is known as *modern algebra*, and it studies general operations on general sets. The most important structures of this type are called *groups*, *rings*, *fields*, *vector spaces*, *modules*, and *algebras*, which are vector spaces whose elements can be multiplied. The most abstract form of algebra, known as *universal algebra*, studies arbitrary unspecified classes

of operations satisfying certain laws, all of which are generalizations of the familiar properties of numbers.

For the sake of perspective, we describe parts of modern algebra briefly in the Epilogue that follows Lesson 11. We will have to invoke some of the concepts of modern algebra toward the end of the story of classical algebra, but for the first nine lessons, we can avoid most of them. The only concept we will make constant use of is that of a *field*, described below. Having now defined our subject matter, we shall henceforth drop the adjective *classical*, with the understanding that when we refer to algebra, we mean the topic of polynomial equations unless we state otherwise.

## 2. Arithmetic and algebra

Most people would probably describe the difference between algebra and arithmetic by saying that in algebra we use letters in addition to numbers. That is a fair way of telling the two subjects apart, but it does not reveal the most important distinction between them. Letters are a convenient notation for recording the processes that we use in algebra, but algebra was being done for some 3000 years before this notation became widespread in the seventeenth century. With a few exceptions such as the Jains in India, who used symbols to represent unknown numbers, the earliest authors wrote their algebra problems in ordinary prose. When you see problems written in prose, it can be more difficult to distinguish between algebra and arithmetic. In both cases, you are given some numbers and asked to find others. What then is the real difference? Let us look at an example to make it clear.

An arithmetic problem:  $3 \times 7 + 36 = ?$

An algebra problem: Solve the equation  $3x + 36 = 57$ .

Let us see what these two problems look like when stated in prose. In the first problem, we are given three numbers (data), namely 3, 7, and 36. We are also given certain processes to perform on these numbers, namely to multiply the first two, then add the third number to the product. We get the answer (57) by following the known rules of arithmetic. Arithmetic amounts to the application of addition, subtraction, multiplication, and division to numbers that are explicitly named.

In the second problem, we are presented with an unknown number. We are told that when it is multiplied by 3 and 36 is added to the product, the result is 57. We must then find the number. As you can see, the biggest difference here is that we are not told what processes we must use in order to find the unknown number. Instead, we are told that some arithmetic was performed on a number, and we are told the result.

Schematically, we are looking at the same underlying process in both cases:

$$(\text{data}), (\text{arithmetic operations}) \quad \rightarrow \quad (\text{result}).$$

In arithmetic we get the data and the operations given to us and must find the result. In algebra, we get the operations and the result and must find the original data.

This difference can be illustrated by analogies from everyday life. The problems that come to us in algebra are a challenge to find concealed numbers. The equations in which they occur are like locked boxes containing valuables. A technique for solving them is like a key to open the box. To take a different analogy, an equation is like a chunk of ore from a mine. The minerals it contains are all jumbled together. It takes a chemist to determine what those minerals are and a metallurgist to separate them so that they can be used. This analogy is better than the first, since chemists and metallurgists study the ways in which minerals combine in order to understand how to separate them again. In the same way, algebraists study the ways in which numbers combine in order to find techniques for separating them, and the study of chemistry or algebra is a perfectly respectable occupation in itself, independently of any minerals or numbers that one may eventually extract from a piece of "ore."

### 3. The "environment" of algebra: Number systems

The data in an equation and its solutions are numbers. But what kind of numbers are they to be? To solve linear problems like the equation  $3x + 36 = 57$  given above, we need only the operations of arithmetic. However, in order to perform these operations, we must have a sufficiently general set of numbers to work with. The positive integers work fine for addition and multiplication. But to make subtraction possible, we need to adjoin zero and the negative integers. Then, to make division (except by zero) possible, we also need to allow all proper and improper fractions. For that reason, the smallest set of numbers that we could possibly consider reasonable would be the *rational numbers* (all fractions, positive and negative, proper and improper). For later reference, we note that a number system in which the four operations of arithmetic are possible, with the exception of division by zero, is called a *field*. For brevity, the four operations of arithmetic are referred to as the *rational operations*. Rational operations can always be performed *within* a field, without adjoining any new elements. In contrast, root extractions are not always possible, and fields must sometimes be enlarged to accommodate them. In fact, the process of enlarging fields by adjoining roots lies at the very heart of the problem of solving equations. Expressions formed using a finite number of rational operations and root extractions are called *algebraic formulas*.

In the present lesson, we shall encounter four fields, the rational numbers, the real numbers, the algebraic numbers, and the complex numbers, all defined below. But there are many others, including some finite fields of considerable interest in algebra, which we shall explore in the problem set below. Let us start with the smallest of these four fields, the rational numbers, which we shall always denote  $\mathbb{Q}$ . These numbers are not sufficient

for solving all equations. To solve an equation like  $x^2 = 10$ , we must be able to extract roots as well, and this operation forces us to consider a larger class of numbers, in which root extractions are possible. We shall refer to these five operations from now on as the *algebraic operations* on numbers.

Allowing root extractions forces us to include certain *irrational* numbers in our set of possible solutions since, for example,  $\sqrt{2}$  is not a rational number. We might even want to solve  $x^2 = -1$ , and so we shall also include *imaginary* and *complex* numbers. A complication arises when we allow root extractions, since every complex number except 0 has exactly two square roots, three cube roots, and so on. For example, the fourth roots of  $-4$  are  $1+i$ ,  $1-i$ ,  $-1+i$ , and  $-1-i$ , where  $i = \sqrt{-1}$ . Thus, when we extract a root, we must either decide which of the possible roots we want, or else live with a symbol representing more than one number. Any complex number that is not a rational number is called an *irrational* number, but the term is most often applied to real numbers that are not rational.

To avoid having to invent new numbers all the time, we need a field that contains the rational numbers and is such that every equation with coefficients in the field will also have a solution in the field. Such a field is called *algebraically closed*. The smallest algebraically closed field is called the set of *algebraic numbers*. This field includes all roots of integers, even roots of negative integers, so that some complex numbers, such as the fourth roots of  $-4$  listed above, are algebraic. To be precise, an algebraic number is any number (real or complex) that satisfies an equation whose coefficients are rational numbers. If you have such an equation, you can multiply it by a common multiple of the denominators of the coefficients and get an equation having the same roots, but with integer coefficients. For example, the equation  $\frac{3}{2}x^2 - \frac{5}{7}x - 0 = 0$  is equivalent to  $21x^2 - 10 = 0$ . Thus, the phrase *rational numbers* in the definition of an algebraic number could have been replaced by the word *integers*.

Algebraic numbers include all numbers that can be formed starting from rational numbers using a finite number of our five classes of operations, for example,  $\sqrt{2} + \sqrt[3]{2}$ . Since there are two square roots of 2 and three cube roots of 2, this expression might represent any of six numbers. Because these six numbers are algebraic, they must be roots of an equation with integer coefficients. You can verify that they are in fact the six roots of the equation

$$x^6 - 6x^3 + 4x^2 + 12x - 4 = 0.$$

**Remark 1.1.** Throughout these lessons, we may use the word *root* to refer to a value of  $x$  that makes a polynomial  $p(x)$  equal to zero ("root of the polynomial") or to a value of  $x$  that makes a polynomial equation  $p(x) = 0$  true ("root of the equation") or to a complex number, some power of which equals a given complex number  $z$  ("root of the complex number  $z$ "), that is, a root of a polynomial  $x^n = z$ , which is the same as a root of the equation  $x^n - z = 0$ .

Obviously, algebraic numbers can easily acquire a very messy appearance, for example,

$$\frac{\sqrt[3]{37} + \sqrt{21}}{4} + \sqrt[3]{19} + \sqrt{\frac{9}{7}}.$$

Thinking of a messy expression like this is a good way to picture a “typical” algebraic number. Because of the ambiguity of taking roots, this expression actually represents 60 different complex numbers! In practice, we would probably choose the simplest of the possible values, which is approximately  $1.36415 + 1.13389i$ .

**Remark 1.2.** A few words of caution are needed here. Although we have just told the reader to think of an algebraic number as a finite expression involving rational numbers, arithmetic operations, and root extractions, numbers of this form are very far from being typical algebraic numbers in an abstract sense. Not every algebraic number can be written as the result of applying a finite number of arithmetic operations and root extractions to rational numbers. In other words, not every algebraic number can be written as a *formula* involving only rational numbers, arithmetical operations, and root extractions. For example, the five roots of the equation  $x^5 - 10x + 2 = 0$  cannot be written this way. This impossibility can be proved using Galois theory, an invention of Évariste Galois (“GAL-wa,” 1811–1832) and the earliest achievement of modern algebra. In Lesson 11, we shall sketch a proof of this impossibility.

As far as algebra itself is concerned, algebraic numbers would be sufficient for all needs. However, many algebra problems arise from applications in geometry, and these are quite likely to involve the number  $\pi$ , which is *not* an algebraic number. Nonalgebraic complex numbers are called *transcendental* numbers, since they “transcend” algebra. Every transcendental number is irrational, but most of the common irrational numbers are algebraic rather than transcendental. Because of the applications in geometry, it is simplest just to take the whole set of real and complex numbers as the set in which we seek solutions of our equations. For our purposes, a real number is a finite or infinite decimal expansion, and a complex number is a number of the form  $a + bi$ , where  $a$  and  $b$  are both real numbers and  $i^2 = -1$ . It happens to be true that every equation with coefficients in this set will also have a solution in the complex numbers. In other words, like the algebraic numbers, the complex numbers form an algebraically closed field, one that is larger than the algebraic numbers.

**Remark 1.3.** Although the complex numbers are used in algebra, and indeed essential in the subject known as algebraic geometry, they are much more geometric than algebraic in nature. The difference between the algebraic and the analytic construction of numbers is well illustrated by the number  $\sqrt{2}$ . In real and complex analysis, this irrational number can be located as the point where the circle through the point  $(1, 1)$  with center

at  $(0, 0)$  intersects the positive real axis. The proof that there actually is a point of intersection involves the order axioms of geometry. For algebraists, a number whose square is 2 is constructed as part of an extension of the field  $\mathbb{Q}$  of rational numbers to a larger field denoted  $\mathbb{Q}(\theta)$ . This larger field consists of formal expressions  $r + s\theta$ , where  $r$  and  $s$  are rational numbers. Addition of such pairs follows the usual algebraic rules, and multiplication is defined by  $(r + s\theta) \times (t + u\theta) = (rt + 2su) + (ru + st)\theta$ . For example,  $(2 + 3\theta) \times (1 + 5\theta) = -28 + 7\theta$ . The field  $\mathbb{Q}$  of rational numbers is identified with a subfield of  $\mathbb{Q}(\theta)$  via the “injection” mapping  $r \mapsto r + 0\theta$ . This injection preserves addition and multiplication, and so makes it reasonable to identify  $\mathbb{Q}$  as a part of  $\mathbb{Q}(\theta)$ . Then the number  $\theta = 0 + 1\theta$  satisfies  $\theta^2 = 2 + 0\theta = 2$ , so that  $\theta$  amounts to a square root of 2.

This algebraic process for constructing  $\sqrt{2}$  is finite, requiring no geometry or approximating processes. Contrast this finiteness with the construction of this number used by analysts. As a real number,  $\sqrt{2}$  requires infinite precision to define, either as the infinitely small point on the intersection of the line and circle mentioned above, or as the infinite decimal expansion  $\sqrt{2} = 1.41421\dots$ , which never repeats and never ends.

The distinction between “finite” algebra and “infinite” or “infinitesimal” (infinitely small) analysis made here is not absolute. As already pointed out, not every algebraic number can be written as a formula involving only a finite number of algebraic operations and rational numbers. Even algebra resorts, at some point, to potentially infinite processes.

**Remark 1.4.** It can be difficult to determine whether a complex number is algebraic. Except for certain artificially constructed examples, the decimal expansion of an irrational number seldom helps to determine whether the number is algebraic or transcendental. Not until the nineteenth century were mathematicians able to prove, for example, that the fundamental constants  $\pi = 3.14159\dots$  and  $e = 2.71828\dots$  are transcendental.

#### 4. Important concepts and principles in this lesson

Before proceeding to the next section, be sure you have a clear picture of each of the following concepts: equation, unknown, coefficient, integer, rational number, rational operation, algebraic number, algebraic formula, transcendental number, real number, and complex number.

As you continue reading, keep in mind the analogies we have introduced here, comparing algebra to the analysis of an ore or the unlocking of a sealed box. Here is another that may help: Doing arithmetic is like cooking; you follow a recipe using specified ingredients processed using available machinery. Doing algebra is like being a food taster; you try to find out what the original ingredients were by looking at the final result.

### 5. Problems and questions

**Problem 1.1.** It is possible to make a field out of as few as two elements, which must necessarily be 0 and 1 and must have the following tables for addition and multiplication:

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

Show that subtraction is the same as addition in this field. That is, the equation  $x = a + b$ , which should have the solution  $x = -a - b$ , actually has the solution  $x = a + b$ . There are only four possible sets of values  $(a, b)$ . Verify this for all four possible choices. In fact, these tables are merely the rules for manipulating even (0) and odd (1) integers, that is, even times odd equals even, and so on.

**Problem 1.2.** Solve the quadratic equations  $x^2 + 1 = 0$  and  $x^2 + x = 0$  in the two-element field just exhibited, and show that the equation  $x^2 - x + 1 = 0$  has no solution at all.

**Problem 1.3.** There is also a field having exactly three elements, which we shall label 1,  $-1$ , and 0. Its addition and multiplication tables are as follows:

+	0	1	$-1$
0	0	1	$-1$
1	1	$-1$	0
$-1$	$-1$	0	1

×	0	1	$-1$
0	0	0	0
1	0	1	$-1$
$-1$	0	$-1$	1

In this field we have the strange-looking rules  $1+1 = -1$  and  $(-1)+(-1) = 1$ . This would look less strange if we adopted a more familiar notation and wrote 2 instead of  $-1$ . If we did that, the strangeness would be expressed in the rules  $1 + 2 = 0$  and  $2 + 2 = 1$ . Show that these rules are merely the common rules for manipulating the remainders when integers are divided by 3. For example, if each of two numbers leaves a remainder of 2 (that is,  $-1$ ) when divided by 3, then their sum and product each leave a remainder of 1. (In other words,  $2 + 2 = 4$  and  $2 \times 2 = 4$ , as usual; only  $4 = 1$  in this case, since the remainder when 4 is divided by 3 is 1.)

**Problem 1.4.** What does subtracting 1 or  $-1$  mean in the three-element field? What does dividing by these elements mean?

**Problem 1.5.** Find all the quadratic equations that can be solved in the three-element field, and write down one that cannot be solved.

**Problem 1.6.** There is a field having exactly four elements, but its arithmetic is not analogous to the arithmetic of the fields with two and three elements. We reserve discussion of this field for Lesson 6 below. Instead, at

this point, we introduce the five-element field, whose elements are  $\bar{2}$ ,  $\bar{1}$ ,  $\bar{0}$ ,  $\bar{1}$ , and  $\bar{2}$ . (Or, if we prefer,  $\bar{0}$ ,  $\bar{1}$ ,  $\bar{2}$ ,  $\bar{3}$ , and  $\bar{4}$ . In any case, it is just the arithmetic of remainders after division by 5.) Its addition and multiplication tables are as follows:

	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{-1}$	$\bar{2}$	$\bar{-2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{-1}$
$\bar{-1}$	$\bar{-1}$	$\bar{0}$	$\bar{-2}$	$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{2}$	$\bar{-2}$	$\bar{1}$	$\bar{-1}$	$\bar{0}$
$\bar{-2}$	$\bar{-2}$	$\bar{-1}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

$\times$	$\bar{0}$	$\bar{1}$	$\bar{-1}$	$\bar{2}$	$\bar{-2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{-2}$
$\bar{-1}$	$\bar{0}$	$\bar{-1}$	$\bar{1}$	$\bar{-2}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{2}$	$\bar{-1}$	$\bar{1}$
$\bar{-2}$	$\bar{0}$	$\bar{-2}$	$\bar{2}$	$\bar{1}$	$\bar{-1}$

What does the fraction  $1/2$  mean in this field? (*Hint:* It should be a solution of the equation  $2x = 1$ .)

**Problem 1.7.** The complex number  $x - iy$  is naturally identified with the point  $(x, y)$  in the plane. Attempts by William Rowan Hamilton (1805-1865) to regard "vectors"  $(x, y, z)$  in three-dimensional space as part of a field, on which rational operations could be performed, were unsuccessful, until he embedded them in a larger four-dimensional space of vectors  $(t, x, y, z)$ . Hamilton named this four-dimensional system *quaternions*. It will be described in the next problem. After that, Josiah Willard Gibbs (1839-1903) was able to distill an algebraic system for three-dimensional vectors by multiplying them as quaternions and projecting them back onto the last three coordinates (the cross product) or the first coordinate (the dot product). For more on vectors in general, see the Epilogue. If  $\alpha = (a_1, a_2, a_3)$  and  $\beta = (b_1, b_2, b_3)$ , their cross product is  $\alpha \times \beta = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ . The vectors  $\alpha$  and  $\beta$  also have an "inner" or "dot" product that is a number rather than a vector:  $\alpha \cdot \beta = a_1b_1 + a_2b_2 + a_3b_3$ .

Verify the following simple facts:

1.  $\alpha \cdot \alpha = a_1^2 + a_2^2 + a_3^2$ . This number is obviously positive unless  $\alpha = (0, 0, 0)$ . Its square root is called the *norm* or *length* or *absolute value* of  $\alpha$  and denoted  $|\alpha| = \sqrt{\alpha \cdot \alpha}$ .
2.  $(\alpha \cdot \beta)^2 \leq (\alpha \cdot \alpha)(\beta \cdot \beta)$ . This inequality is called the *Schwarz inequality* after Hermann Amundus Schwarz (1843-1921). This is obvious if  $\alpha = (0, 0, 0)$ . In all other cases, consider the vector  $\gamma = (\alpha \cdot \beta)\alpha - (\alpha \cdot \alpha)\beta$ , and use the inequality  $\gamma \cdot \gamma \geq 0$ .
3. The angle  $\theta$  between  $\alpha$  and  $\beta$  is defined to be

$$\theta = \arccos \left( \frac{\alpha \cdot \beta}{|\alpha||\beta|} \right).$$

In other words,  $\alpha \cdot \beta = |\alpha||\beta| \cos \theta$ . Then  $\alpha$  is perpendicular to  $\beta$  if and only if  $\alpha \cdot \beta = 0$ .

4. The cross product is anticommutative, that is,  $\beta \times \alpha = -\alpha \times \beta$ . In particular,  $\alpha \times \alpha = \mathbf{0} = (0, 0, 0)$ .
5.  $|\alpha \times \beta|^2 + (\alpha \cdot \beta)^2 = |\alpha|^2 |\beta|^2$ .

6.  $\alpha \times \beta$  is perpendicular to each of its two factors. In fact, if  $n$  is of unit length and perpendicular to both  $\alpha$  and  $\beta$ , then  $\alpha \times \beta = +|\alpha||\beta|\sin\theta n$ . (Transpose one term in the preceding equation to the other side in order to conclude that  $|\alpha \times \beta| = |\alpha||\beta|\sin\theta$ .)

**Problem 1.8.** We can identify the real number  $a$  with the element  $(a, 0, 0, 0)$  in four-dimensional space. And we can identify the vector  $\alpha = (a_1, a_2, a_3)$  with the element  $(0, a_1, a_2, a_3)$  in four-dimensional space. In that way, we can think of a general quaternion  $A = a + \alpha = (a, a_1, a_2, a_3)$  as a formal sum of a number and a vector. Adding two quaternions  $A = a + \alpha$  and  $B = b + \beta$  is trivial:  $A + B = (a + b) + (\alpha + \beta)$ . Multiplying them is less trivial. It took Hamilton some time to work out the proper rules for multiplying elements of four-dimensional spaces. (As we mentioned, Gibbs' work, which we are using to introduce this topic, actually came later.) The proper definition turns out to be  $AB = (ab - \alpha \cdot \beta) + (a\beta + b\alpha + \alpha \times \beta)$ . Notice that  $AB$  is in general different from  $BA$ , since the cross product is antisymmetric.

Show that 1, identified with the quaternion  $\mathbf{1} = (1, 0, 0, 0)$ , has the property  $\mathbf{1}A = A\mathbf{1} = A$  for all quaternions  $A$ .

**Problem 1.9.** Although the order of multiplication makes a difference for quaternions, they do resemble complex numbers in many ways. Quaternions have a real part and a vector part, whereas complex numbers have a real part and an imaginary part. The vector part of a quaternion behaves something like an imaginary number, since if  $a = 0$ , you find that  $A^2 = (0 + \alpha)(0 + \alpha) = (-|\alpha|^2) + \mathbf{0}$ , which is identified with the negative real number  $-|\alpha|^2$ . In other words, each vector  $\alpha$  can be regarded as the square root of the negative of the square of its length.

Show that real numbers commute with all quaternions. That is, the real number  $a$ , identified with the quaternion  $A_0 = a + \mathbf{0}$ , has the property that  $A_0B = BA_0$  for all quaternions  $B$ .

**Problem 1.10.** Like a complex number, the quaternion  $A = a + \alpha$  has a "conjugate"  $\bar{A} = a - \alpha$ . Show that  $AA = a^2 + |\alpha|^2 = a^2 + a_1^2 + a_2^2 + a_3^2$ . We shall write  $|A| = \sqrt{AA} = \sqrt{a^2 + a_1^2 + a_2^2 + a_3^2}$ . Since real numbers commute with all quaternions, it makes sense to define the reciprocal of the quaternion  $A$  as

$$\frac{1}{A} = \frac{\bar{A}}{|A|^2}.$$

Notice that the quotient  $B/A$  isn't well defined. This symbol could mean either  $(1/A)B$  or  $B(1/A)$ , and these two quaternions are in general not the same. Let  $A = (1, 0, 0, 2)$  and  $B = (0, 0, 3, 0)$ . What are the two possible interpretations of  $B/A$ ?

**Problem 1.11.** Describe several different ways of solving the plank-cutting problem of Example 1.1.

**Question 1.1.** Here are some questions of practical use in everyday life. Which of them pose arithmetic problems, and which pose algebra problems?

1. Looking at a stack of current household bills to be paid, how much money must you have in the bank or on hand in order to pay them?
2. With 60% of your grade in a course already determined and your current average at 85%, what average must you maintain for the remaining 40% of the course to ensure a semester average of 90%?
3. The distance  $s$  (in meters) that an object falls in  $t$  seconds, starting from rest and neglecting air resistance, is given by the formula  $s = 4.9t^2$ . How far will an object fall in 7 seconds?
4. Still referring to the formula  $s = 4.9t^2$ , how long will it take an object to fall 120 meters?

**Question 1.2.** Why is there no field having six elements?

### 6. Further reading

H. Behrke, F. Bachmann, K. Fladt, and W. Süß, *Fundamentals of Mathematics*, Vol. 1, translated by S. H. Gould, The MIT Press, Cambridge, MA, 1974.

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