CHAPTER ONE

Notations and Mathematical Preliminaries

1.1 NOTATIONS AND ABBREVIATIONS

The notations and abbreviations used in the book are summarized here for ease of reference.

$D^{(\alpha)} f = f^{\alpha}(t) := df^{\alpha}(t)/dt^{\alpha}$
$\bar{f}$—complex conjugate of f
$\hat{f} := \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$, Fourier transform of $f(t)$
$f(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\,d\omega$, inverse Fourier transform of $\hat{f}(\omega)$
$\|f\|$—norm of a function
$f * g$—convolution
$\langle f, h\rangle := \int \overline{f(t)}h(t)\,dt$, inner product
$f_n = O(n)$-order of n, $\exists C$ such that $f_n \leq Cn$
$\mathcal{C}$—complex
N—nonnegative integers
R—real number
R^n—real numbers of size n
Z—integers
Z^+—positive integers
$L^2(R)$—functional space consisting finite energy functions $\int |f(t)|^2\,dt < +\infty$
$L^p(R)$—function space that $\int |f(t)|^p\,dt < +\infty$
$l^2(Z)$—finite energy series $\sum_{n=-\infty}^{\infty} |a_n|^2 < +\infty$
Ω—set
$H^s(\Omega) := W^{s,2}(\Omega)$-Sobolev space equipped with inner product of
$\langle u, v\rangle_{s,2} := \sum_{|\alpha|\leq s}\int_{\Omega} \overline{D^{\alpha}u}D^{\alpha}v\,d\Omega$

$V \oplus W$—direct sum
$V \otimes W$—tensor product
∇f—gradient
$\vec{H}, \vec{E}$—vector fields
$\nabla \times \vec{H}$—curl
$\nabla \cdot \vec{E}$—divergence
$\lfloor \alpha \rfloor$—largest integer $m \le \alpha$
$\delta_{m,n}$—Kronecker delta
$\delta(t)$—Dirac delta
$\chi[a, b]$—characteristic function, which is 1 in $[a, b]$ and zero outside
$\square$—end of proof
$\exists$—exist
$\forall$—any
iff—if and only if
a.e.—almost everywhere
d.c.—direct current
o.n.—orthonormal
o.w.—otherwise

1.2 MATHEMATICAL PRELIMINARIES

This chapter is arranged here to familiarize the reader with the mathematical notation, definitions and theorems that are used in wavelet literature and in this book. Important mathematical concepts are briefly reviewed. In most cases no proof is given. For more detailed discussions or in depth studies, readers are referred to the corresponding references [1–5].

Readers are suggested to skip this chapter in their first reading. They may then return to the relevant sections of this chapter if unfamiliar mathematical concepts present themselves during the course of the book.

1.2.1 Functions and Integration

A function $f(t)$ is called integrable if

$$\int_{-\infty}^{\infty} |f(t)|\, dt < +\infty, \tag{1.2.1}$$

and we say that $f \in L^1(R)$.

Two functions $f_1(t)$ and $f_2(t)$ are equal in $L^1(R)$ if

$$\int_{-\infty}^{\infty} |f_1(t) - f_2(t)|\, dt = 0.$$

This implies that $f_1(t)$ and $f_2(t)$ may differ only on a set of points of zero measure. The two functions f_1 and f_2 are almost everywhere (a.e.) equal.

Fatou Lemma. Let $\{f_n\}_{n\in N}$ be a set of positive functions. If

$$\lim_{n\to\infty} f_n(t) = f(t)$$

almost everywhere, then

$$\int_{-\infty}^{\infty} f(t)\,dt \le \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(t)\,dt.$$

This lemma provides an inequality when taking a limit under the Lebesgue integral for positive functions.

Lebesgue Dominated Convergence Theorem. Let $f_k(t) \in L(E)$ for $k = 1, 2, \ldots,$ and

$$\lim_{k\to\infty} f_k(t) = f(t) \quad \text{a.e.}$$

If there exists an integrable function $F(t)$ such that

$$|f_k(t)| \le F(t) \quad \text{a.e.}, \quad k = 1, 2, \ldots,$$

then

$$\lim_{k\to\infty} \int_E f_k(t)\,dt = \int_E f(t)\,dt.$$

This theorem allows us to exchange the limit with integration.

Fubini Theorem. If

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t_1, t_2)\,dt_1 \right) dt_2 < \infty,$$

then

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t_1, t_2)\,dt_1\,dt_2 &= \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{\infty} f(t_1, t_2)\,dt_1 \\ &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} f(t_1, t_2)\,dt_2. \end{aligned}$$

This theorem provides a sufficient condition for commuting the order of the multiple integration.

1.2.2 The Fourier Transform

The Fourier transform pair is defined as

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt,$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\, d\omega.$$

Rigorously speaking, the Fourier transform of $f(t)$ exists if the Dirichlet conditions are satisfied, that is,

(1) $\int_{-\infty}^{\infty} |f(t)|\, dt < +\infty$, as in (1.2.1).
(2) $f(t)$ has a finite number of maxima and minima within any finite interval, and any discontinuities of $f(t)$ are finite. There are only a finite number of such discontinuities in any finite interval.

All functions satisfying (1.2.1) form a functional space L^1. A weaker condition for the existence of the Fourier transform of $f(t)$, in replace of (1.2.1), is given as

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt < +\infty. \tag{1.2.2}$$

All functions satisfying (1.2.2) form a functional space L^2.

When the Dirichlet conditions are satisfied, the inverse Fourier transform converges to $f(t)$ if $f(t)$ is continuous at t, or to

$$\frac{f(t^+) + f(t^-)}{2}$$

if $f(t)$is discontinuous at t. When $f(t)$ has infinite energy, its Fourier transform may be defined by incorporating generalized functions. The resultant is called the generalized Fourier transform of the original function.

1.2.3 Regularity

Lipschitz Regularity. If a function $f(t)$ has a singularity at $t = v$, this implies that $f(t)$ is not differentiable at v. Lipschitz exponent at v characterizes the singularity behavior.

The Taylor expansion relates the differentiability of a function to a local polynomial approximation. Suppose that f is m times differentiable in $[v - h, v + h]$. Let p_v be the Taylor polynomial in the neighborhood of v:

$$p_v(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(v)}{k!}(t - v)^k.$$

Then the error

$$|\varepsilon_v(t)| \le \frac{|t-v|^m}{m!} \sup_{u\in[v-h,v+h]} |f^{(m)}(u)|$$

where

$$t \in [v-h, v+h], \quad \varepsilon_v(t) := f(t) - p_v(t).$$

The Lipschitz regularity refines the upper bound on the error $\varepsilon_v(t)$ with noninteger exponents. Lipschitz exponents are also referred to as Hölder exponents.

Definition 1 (Lipschitz). A function $f(t)$ is pointwise Lipschitz $\alpha \ge 0$ at $t = v$, if there exist $M > 0$ and a polynomial $p_v(t)$ of degree $m = \lfloor\alpha\rfloor$ such that

$$\forall t \in R, \quad |f(t) - p_v(t)| \le M|t-v|^{\alpha}. \tag{1.2.3}$$

Definition 2. A function $f(t)$ is uniformly Lipschitz α over $[a, b]$ if it satisfies (1.2.3) for all $v \in [a, b]$ with a constant M independent of v.

Definition 3. The Lipschitz regularity of $f(t)$ at v or over $[a, b]$ is the *sup* of the α such that $f(t)$ is Lipschitz α.

Theorem 1. A function $f(t)$ is bounded and uniform Lipschitz α over R if

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|(1+|\omega|^{\alpha})\, d\omega < +\infty. \tag{1.2.4}$$

If $0 \le \alpha < 1$, then $p_v(t) = f(v)$ and the Lipschitz condition reduces to

$$\forall t \in R, \quad |f(t) - f(v)| \le M|t-v|^{\alpha}.$$

Here the function is bounded but discontinuous at v, and we say that the function is Lipschitz 0 at v.

Proof. When $0 \le \alpha < 1$, it follows $m := \lfloor\alpha\rfloor = 0$, and $p_v(t) = f(v)$.

The uniform Lipschitz regularity implies that $\exists M > 0$ such that

$$\forall (t, v) \in R^2.$$

We need to have

$$\frac{|f(t) - f(v)|}{|t-v|^{\alpha}} \le M.$$

Since

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t}\, d\omega, \\
\frac{|f(t) - f(v)|}{|t - v|^{\alpha}} &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \hat{f}(\omega) \left[\frac{e^{i\omega t}}{|t - v|^{\alpha}} - \frac{e^{i\omega v}}{|t - v|^{\alpha}} \right] d\omega \right| \\
&\le \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)| \frac{|e^{i\omega t} - e^{i\omega v}|}{|t - v|^{\alpha}}\, d\omega.
\end{aligned}
$$

(1) For $|t - v|^{-1} \le |\omega|$,

$$
\frac{|e^{i\omega t} - e^{i\omega v}|}{|t - v|^{\alpha}} \le \frac{2}{|t - v|^{\alpha}} \le 2|\omega|^{\alpha}.
$$

(2) For $|t - v|^{-1} \ge |\omega|$,

$$
|e^{i\omega t} - e^{i\omega v}| = \left| i\omega(t - v) - \frac{\omega^2}{2!}(t - v)^2 - i\frac{(t - v)^3}{3!} + - \cdots \right|.
$$

On the right-hand side of the equation above, the imaginary part

$$
I = \omega(t - v) - \frac{[\omega(t - v)]^3}{3!} + \frac{[\omega(t - v)]^5}{5!} - + \cdots \le \omega(t - v),
$$

and the magnitude of the real part

$$
R = \left\{ \frac{[\omega(t - v)]^2}{2!} - \frac{[\omega(t - v)]^4}{4!} + - \cdots \right\} \le \frac{[\omega(t - v)]^2}{2!}.
$$

Thus

$$
|(t - v)\omega| \le 1 \quad \text{and} \quad [(t - v)\omega]^2 \le |(t - v)\omega|
$$

and

$$
\begin{aligned}
|e^{i\omega t} - e^{i\omega v}| &\le \left| i\omega(t - v) + \frac{[\omega(t - v)]^2}{2!} \right| \\
&= \sqrt{[\omega(t - v)]^2 + \frac{\omega^4 (t - v)^4}{4}} \\
&\le |2\omega(t - v)|.
\end{aligned}
$$

Hence

$$
\frac{|e^{i\omega t} - e^{i\omega v}|}{|t - v|^{\alpha}} \le \frac{2|\omega||t - v|}{|t - v|^{\alpha}} \le 2|\omega|^{\alpha}.
$$

Combining (1) and (2) yields

$$\frac{|f(t)-f(v)|^2}{|t-v|^\alpha} \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} 2|\hat{f}(\omega)|\,|\omega|^\alpha\,d\omega := M.$$

It can be verified that if

$$\int_{-\infty}^{\infty} |\hat{f}(\omega)|[1+|\omega|^p]\,d\omega < \infty,$$

then $f(t)$ is p times continuously differentiable. Therefore, if

$$\int_{-\infty}^{\infty} \hat{f}(\omega)[1+|\omega|^\alpha]\,d\omega < \infty,$$

then $f^{(m)}(t)$ is uniformly Lipschitz $\alpha - m$, and hence $f(t)$ is uniformly Lipschitz α, where $m = \lfloor \alpha \rfloor$. □

1.2.4 Linear Spaces

Linear Space. A linear space H is a nonempty set. Let $\mathcal{C}$ be complex. H is called a complex linear space if

(1) $x + y = y + x$.
(2) $(x + y) + z = x + (y + z)$.
(3) There exists a unique element $\theta \in H$ such that for $\forall x \in H, x + \theta = \theta + x$.
(4) For $\forall x \in H$, there exists a unique $-x$ such that $x + (-x) = \theta$.

In addition we define scalar multiplication $\forall(\alpha, x) \in \mathcal{C} \times H$ such that

(1) $\alpha(\beta x) = (\alpha\beta)x, \forall\alpha, \beta \in \mathcal{C}, \forall x \in H$.
(2) $1x = x$.
(3) $(\alpha + \beta)x = \alpha x + \beta x, \forall\alpha, \beta \in \mathcal{C}, \forall x \in H$.
$\alpha(x + y) = \alpha x + \alpha y, \forall\alpha \in \mathcal{C}, \forall x, y \in H$.

Norm of a Vector

Definition. Mapping of $\| x \|: R^n \to R$ is called the norm of x on R^n iff

(1) $\| x \| \geq 0, \forall x \in R^n$.
(2) $\| \alpha x \| = |\alpha|\, \| x \|, \forall\alpha \in R, x \in R^n$.
(3) $\| x + y \| \leq \| x \| + \| y \|, \forall x, y \in R^n$.
(4) $\| x \| = 0 \Longleftrightarrow x = 0$.

Let $x = (x_1, x_2, \ldots, x_n)^T \in R^n$. The following are commonly used norms:

$$\| x \|_\infty = \max_i |x_i|, \qquad \ell^\infty \text{ norm,}$$

$$\| x \|_1 = \sum_{i=1}^{n} |x_i|, \qquad \ell^1 \text{ norm,}$$

$$\| x \|_2 = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}, \qquad \ell^2 \text{ norm,}$$

$$\| x \|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}, \qquad \ell^p \text{ norm.}$$

1.2.5 Functional Spaces

Metric, Banach, Hilbert, and Sobolev spaces are functional spaces. A functional space is a collection of functions that possess a certain mathematical structure pattern.

Metric Space. A metric space H is a nonempty set that defines the distance of a real-valued function $\rho(x, y)$ that satisfies:

(1) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0$ iff $x = y$.
(2) $\rho(x, y) = \rho(y, x)$.
(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y), \forall x, y, z \in H$.

Banach Space. Banach space is a vector space H that admits a norm, $\| \cdot \|$, that satisfies:

(1) $\forall f \in H, \| f \| \geq 0$ and $\| f \| = 0$ iff $f = 0$.
(2) $\forall \alpha \in \mathcal{C}, \| \alpha f \| = |\alpha| \, \| f \|$.
(3) $\| f + g \| \leq \| f \| + \| g \|, \forall f, g \in H$.

These properties of norms are similar to those of distance, except the homogeneity of (2) is not required in defining a distance. The convergence of $\{f_n\}_{n \in N}$ to $f \in H$ implies that $\lim_{n \to \infty} \| f_n - f \| = 0$ and is denoted as $\lim_{n \to \infty} f_n = f$.

To guarantee that we remain in H when taking the limits, we define the Cauchy sequences. A sequence $\{f_n\}_{n \in N}$ is a Cauchy sequence if for $\forall \varepsilon > 0$, there exist n and m large enough such that $\| f_m - f_n \| < \varepsilon$. The space H is said to be complete if every Cauchy sequence in H converges to an element of H. A complete linear space equipped with norm is called the Banach space.

Example 1 Let S be a collection of sequences $x = (x_1, x_2, \ldots, x_n, \ldots)$. We define addition and multiplication naturally as

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n, \ldots),$$
$$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n, \ldots),$$

and define distance as

$$\rho(x, y) = \sum \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$

It can be verified that such a space S is not a Banach space, because $\rho(x, y)$ does not satisfy the homogeneous condition of the norm.

Example 2 For any integer p we define over discrete sequence f_n the norm

$$\|f\|_p = \left[\sum_{n=-\infty}^{\infty} |f_n|^p \right]^{1/p}.$$

The space $\ell^p = \{f : \|f\|_p < \infty\}$ is a Banach space with norm $\|f\|_p$.

Example 3 The space $L^p(R)$ is composed of measurable functions f on R that

$$\| f \|_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p \right\}^{1/p} < \infty.$$

The space $L^p(R) = \{f : \| f \|_p < \infty\}$ is a Banach space.

Hilbert Space. A Hilbert space is an inner product space that is complete. The inner product satisfies:

(1) $\langle \alpha f + \beta g, h \rangle = \overline{\alpha} \langle f, g \rangle + \overline{\beta} \langle g, h \rangle$ for $\alpha, \beta, \in \mathcal{C}$ and $f, g, h \in H$.
(2) $\langle f, g \rangle = \overline{\langle g, f \rangle}$.
(3) $\langle f, f \rangle \geq 0$ and $\langle f, f \rangle = 0$ iff $f = 0$. One may verify that

$$\| f \| = \langle f, f \rangle^{1/2}$$

is a norm.
(4) The Cauchy–Schwarz inequality states that

$$|\langle f, g \rangle| \leq \| f \| \| g \|,$$

where the equality is held iff f and g are linearly dependent.

In a Banach space the norm is defined, which allows us to discuss the convergence. However, the angles and orthogonality are lacking. A Hilbert space is a Banach space equipped with an inner product.

1.2.6 Sobolev Spaces

The Sobolev space is a functional space, and it could have been listed in the previous subsection. However, we have placed it in a separate subsection because of its contents and role in the text.

On many occasions involving differential operators, it is convenient to incorporate the L^p norms of the derivative of a function into a Banach norm. Consider the functions in the class $C^\infty(\Omega)$. For any number $p \geq 1$ and number $s \geq 0$, let us take the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|_{s,p} = \left\{ \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^p}^p \right\}^{1/p}. \tag{1.2.5}$$

The resulting Banach space is called the Sobolev space $W^{s,p}(\Omega)$. For $p = 2$ we denote $W^s(\Omega) = W^{s,2}(\Omega)$, which is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{s,2} = \sum_{|\alpha| \leq s} \int_\Omega \overline{D^\alpha u} \cdot D^\alpha v \, dx.$$

Sometimes $W^s(R)$ is also denoted as $H^s(R)$. Note that the differentiation in (1.2.5) can be of a noninteger.

Recall that the Fourier transform of the derivative $f'(t)$ is $i\omega \hat{f}(\omega)$. The Plancherel–Parseval formula proves that $f'(t) \in L^2(R)$ if

$$\int_{-\infty}^{\infty} |f'(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^2 |\hat{f}(\omega)|^2 < +\infty.$$

This expression can be generalized for any $s > 0$,

$$\int_{-\infty}^{\infty} |\omega|^{2s} |\hat{f}(\omega)|^2 \, d\omega < +\infty$$

if $f \in L^2(R)$ is s times differentiable.

Considering the summation nature of (1.2.5), we can write the more precise expression of Sobolev space in the Fourier domain as

$$\int_{-\infty}^{\infty} (1 + \omega^2)^s |f(\omega)|^2 \, d\omega < +\infty.$$

For $s > n + \frac{1}{2}$, f is n times continuously differentiable. The Sobolev space H^α, $\alpha \in R$ consists of functions $f(t) \in S'$ such that

$$\int_{-\infty}^{\infty} \hat{f}(\omega)(1 + \omega^2)^\alpha \, d\omega < \infty.$$

For $\alpha = 0$, the H^α reduces to $L^2(R)$. For $\alpha = 1, 2, \ldots$, H^α is composed of ordinary $L^2(R)$ functions that are $(\alpha - 1)$ times differentiable and whose αth derivative are in $L^2(R)$. For $\alpha = -1, -2, \ldots,$ H^α contains the $-\alpha$th derivatives of $L^2(R)$ and all distributions with point support of order $< \alpha$.

It can be seen $H^\alpha \supset H^\beta$ when $\alpha > \beta$. The inner product of $f, g \in H^\alpha$ is

$$\langle f, g\rangle_\alpha = \frac{1}{2\pi}\int \overline{\hat{f}(\omega)}\hat{g}(\omega)(1+\omega^2)^\alpha \, d\omega$$

and is complete with respect to this inner product. Therefore it is a Hilbert space.

1.2.7 Bases in Hilbert Space *H*

Orthonormal Basis. A sequence $\{f_n\}_{n\in N}$ in a Hilbert space H is orthonormal if

$$\langle f_m, f_n\rangle = \delta_{m,n}.$$

If for $f \in H$ there exist α_n such that

$$\lim_{N\to\infty} \|f - \sum_{n=0}^{N} \alpha_n f_n\| = 0,$$

then $\{f_n\}_{n\in N}$ is called an orthogonal basis of H.

For an orthonormal basis we require $\|f_n\| = 1$. A Hilbert space that admits an orthogonal basis is said to be separable. The norm of $f \in H$ is

$$\|f\|^2 = \sum_{n=0}^{\infty} |\langle f, f_n\rangle|^2$$

Riesz Basis. Let $\{f_n\}$ be linear independent and complete in $L^2(a, b)$, meaning that the closed linear span of $\{f_n\}$ is $L^2(a, b)$. The set is called a Riesz basis if there exist $A > 0$ and $B > 0$ such that

$$A\sum_i |c_i|^2 \le \|\sum_i c_i f_i\|^2 \le B\sum_i |c_i|^2 \tag{1.2.6}$$

for each sequence $\{c_i\}$ of complex numbers. The Riesz representation theorem guarantees the existence of the dual $\{\tilde{f}_n\}$ in $L^2(a, b)$ such that:

(1) $\{\tilde{f}_n\}$ is the unique biorthogonal sequence to $\{f_n\}$; namely $\langle f_m, \tilde{f}_n\rangle = \delta_{m,n}$.
(2) If $\{c_n\} \in \ell^2$, then $\sum_n c_n f_n$ converges in $L^2(a, b)$.
(3) For each $f \in L^2(a, b)$, $\{\langle f, \tilde{f}_n\rangle\} \in \ell^2$.
(4) For each $f \in L^2(a, b)$,

$$f = \sum_{i=0}^{\infty} \langle f, \tilde{f}_i\rangle f_i = \sum_{i=0}^{\infty} \langle f, f_i\rangle \tilde{f}_i.$$

A Riesz basis of a separable Hilbert space H is a basis that is close to being orthogonal. The right inequality in (1.2.6) is essential. It prevents the expansion from blowing up. The left inequality in (1.2.6) is important too, since it ensures the existence of the inverse.

1.2.8 Linear Operators

In computational electromagnetics, the method of moments and finite element method are based on linear operations. An operator T from a Hilbert space H_1 to another Hilbert space H_2 is linear if

$$\forall \alpha_1, \alpha_2 \in \mathcal{C},\ \forall f_1, f_2 \in H_1, \quad T(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 T(f_1) + \alpha_2 T(f_2).$$

Sup Norm. The sup operator norm of T is defined as

$$\|T\|_S = \sup_{f \in H_1} \frac{\|Tf\|}{\|f\|}. \tag{1.2.7}$$

If this norm is finite, then T is continuous; namely $\|Tf_1 - Tf_2\|$ becomes arbitrarily small if $\|f_1 - f_2\|$ is sufficiently small.

Adjoint. The adjoint of T is the operator T^a from H_2 to H_1 such that for any $f_1 \in H_1$ and $f_2 \in H_2$

$$\langle Tf_1, f_2 \rangle = \langle f_1, T^a f_2 \rangle.$$

When T is defined from H into itself, it is self-adjoint if $T = T^a$. A nonzero vector $f \in H$ is a called an *eigenvector* if there exists an *eigenvalue* $\lambda \in \mathcal{C}$ such that

$$Tf = \lambda f.$$

In a finite-dimensional Hilbert space, meaning that Euclidean space, a self-adjoint operator is always diagonalized by an orthogonal basis $\{e_n\}_{0 \le n < N}$ of eigenvectors

$$Te_n = \lambda_n e_n.$$

For a self-adjoint operator T, the eigenvalues λ_n are real, and for any $f \in H$

$$Tf = \sum_{n=0}^{N-1} \langle Tf, e_n \rangle e_n = \sum_{n=0}^{N-1} \lambda_n \langle f, e_n \rangle e_n.$$

In an infinite-dimensional Hilbert space, the previous result can be generalized in terms of the spectrum of the operator, which must be manipulated with caution.

Orthogonal Projector. Let V be a subspace of H. A *projector* P_V on V is a linear operator that satisfies $\forall f \in H, \quad P_V f \in V$ and $\forall f \in V, \quad P_V u = f$.

The projector P_V is *orthogonal* if

$$\forall f \in H, \forall g \in V, \quad \langle f - P_V f, g \rangle = 0.$$

The following properties are often used in the text:

Property 1. If P_V is *a projector* on V, then the following statements are equivalent:

(1) P_V is *orthogonal.*
(2) P_V is *self-adjoint.*
(3) $\|P_V\|_S = 1$.
(4) $\forall f \in H$, $\|f - P_V f\| = \min_{g \in \mathbf{V}} \|f - g\|$.

If $\{e_n\}_{n\in N}$ is an orthogonal basis of V, then

$$P_V f = \sum_{n=0}^{+\infty} \frac{\langle f, e_n \rangle}{\|e_n\|^2} e_n.$$

If $\{e_n\}_{n\in N}$ is a Riesz basis of V and $\{\tilde{e}_n\}_{n\in N}$ is the biorthogonal basis, then

$$P_V f = \sum_{n=0}^{+\infty} \langle f, e_n \rangle \tilde{e}_n = \sum_{n=0}^{+\infty} \langle f, \tilde{e}_n \rangle e_n.$$

Density and Limit. A space V is *dense* in H if for any $f \in H$ there exist $\{f_m\}_{m\in N}$ with $f_m \in V$ such that

$$\lim_{m\to+\infty} \|f - f_m\| = 0.$$

Let $\{T_n\}_{n\in N}$ be a sequence of linear operators from H to H. Such a sequence *converges* weakly to a linear operator T_∞ if

$$\forall f \in H, \quad \lim_{n\to+\infty} \|T_n f - T_\infty f\| = 0.$$

To find the limit of operators it is preferable to work in a well chosen subspace $V \subset H$ which is dense. The density and limit are justified by the property below.

Property 2 (Density). Let V be a *dense subspace* of H. Suppose that there exists C such that $\|T_n\|_S \leq C$ for all $n \in N$. If

$$\forall f \in V, \quad \lim_{n\to+\infty} \|T_n f - T_\infty f\| = 0,$$

then

$$\forall f \in H, \quad \lim_{n\to+\infty} \|T_n f - T_\infty f\| = 0.$$

For numerical computations, an operator is often discretized into a matrix. Only then digital computers can be utilized.

Norm of a Matrix. For a matrix $A \in R^{n\times n}$, the norm of A is defined, similarly to (1.2.7), as

$$\| A \| = \max_{x\neq 0} \left\{ \frac{\| Ax \|}{\| x \|} \right\}.$$

In particular, the commonly used norms are as follows:

(1) The column norm (ℓ^1 norm)

$$\| A \|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|.$$

(2) The row norm (ℓ^∞ norm)

$$\| A \|_\infty = \max_i \{\|a_{i\cdot}\|_1\} = \max_i \sum_{j=1}^{n} |a_{i,j}|.$$

(3) The spectral norm (ℓ^2 norm)

$$\|A\|_2 = (\lambda_{A^T A})^{1/2},$$

where $\lambda_{A^T A}$ is the maximum eigenvalue of $A^T A$.

(4) The Frobenius norm

$$\|A\|_F = \left(\sum_{j=1}^{n} \sum_{i=1}^{n} |a_{i,j}|^2 \right)^{1/2} = [\mathrm{tr}\,\{A^T A\}]^{1/2}.$$

BIBLIOGRAPHY

[1] R. L. Wheedan and A. Zygmund, *Measure and Integration*, Marcel Dekker, New York, 1977.

[2] A.N. Kolmogorov and S.V. Fomin, *Introductory Real Analysis*, Dover, New York, 1970.

[3] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Dover Publication, Inc. New York, 1990.

[4] M. S. Berger, *Nonlinear and Functional Analysis*, Academic Press, New York, 1977.

[5] S. Mallat, *A Wavelet Tour of Signal Processing*, Academic Press, New York, 1998.