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INTRODUCTION

Mechanics studies the theory of forces and their interactions. It is the pillar of modern sciences. Many of the great scientists in history have been mechanicians including, for example, Aristotle, Archimedes, Leonardo da Vinci, Kepler, and Newton.

Continuum mechanics is a relatively young branch of mechanics. It studies the deformation of bodies (solid or fluid) under forces or stresses. Although the basic foundation of continuum mechanics was laid by Galileo, it was during the late eighteenth and the early nineteenth centuries that modern theories of continuum mechanics were gradually developed by Laplace, Fourier, Coriolis, Lagrange, Hamilton, Navier, and Cauchy, among others. It was during this period and up until the early 1900s that continuum mechanics enjoyed its most rapid development into maturity. By the mid-1900s, theories of continuum mechanics had been established upon vigorous mathematics.

One of the most successful stories of continuum mechanics is the development and application of fracture mechanics. In 1913, C.E. Inglis looked at a thin plate of glass with an elliptical hole in the middle in a new and different way (Fig. 1.1). The plate was pulled at both ends perpendicular to the ellipse. He found that the stress at point A is given by

$$\sigma_A = \sigma\left(1 + \frac{2a}{b}\right).$$

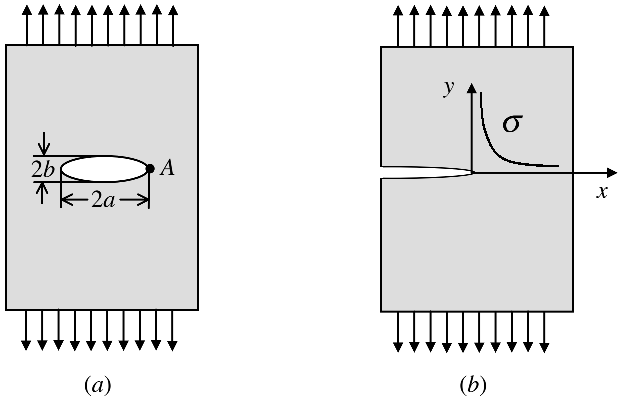

Figure 1.1 (*a*) Inglis' work on stress concentration near an elliptical hole. (*b*) Griffith's work on stress concentration near a sharp crack tip.

In other words, the stress at the tip of the elliptical hole can be much larger than the stress applied. In the 1920s, A.A. Griffith extended Inglis' work. He found that the stress at the ends of the crack approaches infinity:

$$\sigma = \frac{K}{\sqrt{2\pi x}}.$$

Griffith also introduced the notion of energy. He said that for a crack to grow, it was necessary for there to be enough potential energy in the system to create the new surface area of the crack. Although he did not know that it takes more than this for a crack to grow, Griffith's idea of fracture criterion laid the foundation for a brand new theory called fracture mechanics, which is one of the most celebrated branches of continuum mechanics in modern history.

More importantly, fracture mechanics brought continuum mechanics and material science together and opened up new opportunities for studying not only deformation of solid bodies but also the failure behavior of solid materials under load. Thus a new field called mechanics of materials emerged. In mechanics of material, we use the vigorous continuum mechanics theories to investigate and study how materials with certain microstructures deform and eventually fail under given loads or stresses.

Micromechanics is a branch of mechanics of materials. It is the most recent development in applying continuum mechanics theories to real materials. The beginning of micromechanics may be traced back to Eshelby's seminal study (Eshelby, 1957). But, the theory of micromechanics was not fully developed into a subject area of its own until the early 1980s. Even though it is still an actively researched area now, the theory of micromechanics has matured enough that several books have been published. Among them, the books by Mura (1986), Nemat-Nasser and Hori (1993), and Krajcinovic (1996) are probably the most comprehensive and influential ones.

The theory of micromechanics of solids describes the scientific concepts, principles, and methodologies for the study of thermomechanical behavior of heterogeneous materials. Although the fundamental equations of micromechanics are based on mechanics of continuum, its applications cover a broad range of thermomechanical behavior of materials including plasticity, fracture, and fatigue, constitutive equations of composites, and polycrystalline materials. For example, it applies the theories of elasticity and plasticity to study imperfections in crystals, inclusions, and inhomogeneities in alloys and composite materials. The objective is to study the macroscopic mechanical behavior of materials from an understanding of their microstructure. This involves the application of continuum mechanics to identifiable small-scale structures and the use of analytical and numerical methods to compute the macroscopic responses. This science-based approach enables us to predict the behavior of new materials without the need for physical experimentation. It provides a powerful tool for engineering design, fabrication, and analysis of a wide range of materials including polycrystalline, composite, geotechnical, biological, and electronic materials. Optimum microstructures can be forecasted rather than found by trial and error. Fracture and fatigue of solids and structures, martensitic transformations, interphases in composites, and dispersion hardening of alloys are examples of the phenomena that are being elucidated and qualified by micromechanics.

1.1 BACKGROUND AND MOTIVATION

As the theory of micromechanics matures, many universities around the world are offering courses on this subject. For the past 15 years, the authors have taught micromechanics classes in their respective institutions. We have always been frustrated by not being able to find an appropriate textbook for the course. Most existing books on this subject are research monographs, primarily for experts and researchers. They can be excellent research tools but not convenient to use as textbooks. Because the theory of micromechanics is still in its infancy, results were obtained by individual researchers using, sometime, very different approaches/methodologies. In order for the students (or first-time learners) to understand the intrinsic connections among different concepts and approaches, a unified approach (including the use of notations) is needed to develop the micromechanics theory. This will allow both the instructor and the students to fully grasp the essence of the theory. Furthermore, instead of collecting and compiling existing results from the literature, we should identify a set of topics that convey the fundamental ideas of micromechanics and focus on these topics. Related topics not covered in the text should be referenced for those who wish to learn more. Exercise problems should be provided for the convenience of the instructor, as well as for those who wish to study the subject on their own. These are the major considerations that motivated us to write this textbook.

1.2 OBJECTIVES

The intent of this book is not to provide a comprehensive collection of results of micromechanics in the literature, nor is it to be a research reference book for relevant publications. Instead, it is intended to be a textbook for graduate and possibly upper-level undergraduate students. It is to provide a teaching tool for an instructor to teach and a learning aid for a beginner to learn (what we believe) the most fundamental ideas and approaches, the basic concepts, principles, and methodologies of micromechanics. To this end, a unified mathematical framework is introduced early on in the book. The rest of the theories will be developed based on this framework in a logical and easily understandable approach. In addition to some new results from the authors' own research, many of the available results in the literature will be derived or re-derived based on this unified mathematical framework. This approach enables the students to follow the various developments of the micromechanics theories. It also helps the students to quickly comprehend and appreciate the wide range of applications of micromechanics.

1.3 ORGANIZATION OF BOOK

The book is organized into 13 chapters. References and/or Suggested Readings are included at the end of each chapter. Some of these references contain certain results not derived but used in the text. Others are listed because they present either alternative approaches for the same problem or provide additional topics related to those discussed in the chapter. We have made great effort to make the book somewhat self-contained. The goal was that a reader with the basic knowledge of continuum mechanics should be able to follow this book without consulting other publications.

Each chapter also contains a set of problems. The students, as well as the instructor, may find these exercises useful. The level of difficulty varies significantly among the problems. The students should not feel discouraged if they cannot solve some of the problems on their first attempt.

The reminder of this chapter presents the most frequently used notations and notation conventions used in this book. In the next chapter, a brief summary of the basic theories of continuum mechanics is presented. The rest of the chapters are grouped into linear theories (Chapters 3–10) and nonlinear theories (Chapter 11–13).

1.4 NOTATION CONVENTIONS

One of the difficulties many people encounter in studying micromechanics is the different kinds of notations used in the literature. For consistency, we will use the following conventions throughout this book, unless otherwise noted.

Index notation for vectors and tensors will be used extensively. Whenever possible, the base letter for a vector (first-order tensor) will be a lowercase italic letter, for a second-order tensor it will be a lowercase Greek letter, and for a fourth-order tensor it will be an uppercase italic letter. For example, u_i represents a vector, ε_{ij} represents a second-order tensor, and L_{ijkl} represents a fourth-order tensor. Exceptions to these rules are certain letters conventionally used for specific physical entities. For example, G_{ij} is for the Green function, which is a second-order tensor.

The summary convention will be used:

$$L_{ijkl}\varepsilon_{kl} \equiv \sum_{l=1}^{3} \sum_{k=1}^{3} L_{ijkl}\varepsilon_{kl}. \tag{1.4.1}$$

Alternately, when it is convenient, the direct (or matrix) notation of vectors and tensors will be used as well. Boldface letters will be used for this purpose. For example, $\mathbf{u}$ represents a vector, $\boldsymbol{\varepsilon}$ represents a second-order tensor, and $\mathbf{L}$ represents a fourth-order tensor. To distinguish tensors with the same base letter a subscript will be used. For example, $\mathbf{L}_1$ and $\mathbf{L}_2$ are used to denote two different fourth-order tensors. Note that these subscripts are entirely different from the subscripts used to represent the components of a tensor. Subscripts on matrix notations do not follow the summation convention, therefore, when a tensor in the matrix notation with a subscript is written in index forms, we will change the subscript to superscript with parenthesis, for example,

$$\mathbf{L}_1 \Leftrightarrow L^{(1)}_{ijkl} \quad \text{and} \quad \mathbf{L}_2 \Leftrightarrow L^{(2)}_{ijkl}.$$

Throughout the book, index and matrix notations will be used interchangeably based on whichever is convenient.

To avoid confusion, we use the following notations to represent the tensor algebraic operations.

Dot product:	$\sigma_{ij}n_j \Leftrightarrow \boldsymbol{\sigma} \cdot \mathbf{n}$	
Double-dot product:	$L_{ijkl}\varepsilon_{kl} \Leftrightarrow \mathbf{L}:\boldsymbol{\varepsilon}$,	$L_{ijkl}T_{klmn} \Leftrightarrow \mathbf{L}:\mathbf{T}$
Dyad:	$m_i n_j \Leftrightarrow \mathbf{m} \otimes \mathbf{n}$,	$n_i m_j \Leftrightarrow \mathbf{n} \otimes \mathbf{m}$

Since dot and double-dot operations will be used extensively, we will, when there is no ambiguity, neglect the dot(s) and simply write, for example, $\mathbf{L}:\boldsymbol{\varepsilon} \Leftrightarrow \mathbf{L}\boldsymbol{\varepsilon}$ and $\sigma \cdot \mathbf{n} \Leftrightarrow \sigma\mathbf{n}$.

A fourth-order tensor, $\mathbf{A}$, is nonsingular if and only there exists a fourth-order tensor, for example, $\mathbf{B}$, such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}. \tag{1.4.2}$$

In this case, $\mathbf{B}$ is the inverse of $\mathbf{A}$, or $\mathbf{A}$ is the inverse of $\mathbf{B}$, that is,

$$\mathbf{B} = \mathbf{A}^{-1} \quad \text{or} \quad \mathbf{A} = \mathbf{B}^{-1}. \tag{1.4.3}$$

An equivalent definition can be given as follows: $\mathbf{A}$ is singular if and only if there exists a second-order tensor $\sigma \neq \mathbf{0}$, such that $\mathbf{A}\sigma = \mathbf{0}$.

A fourth-order isotropic tensor can be written as

$$A_{ijkl} = a\delta_{ij}\delta_{kl} + b(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}). \tag{1.4.4}$$

For convenience, we introduce the following two fourth order tensors:

$$I^h_{ijkl} = \tfrac{1}{3}\delta_{ij}\delta_{kl}, \qquad I^d_{ijkl} = \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}), \tag{1.4.5}$$

so that the fourth-order isotropic tensor given in (1.4.4) can be written as

$$\mathbf{A} = 3a\mathbf{I}^h + 2b\mathbf{I}^d. \tag{1.4.6}$$

It can be easily shown that, if $\boldsymbol{\sigma}$ is a second-order tensor, then

$$\mathbf{A}\boldsymbol{\sigma} = 3a\sigma\mathbf{I} + 2b\boldsymbol{\sigma}', \tag{1.4.7}$$

where $\sigma = \sigma_{kk}$ is the spherical part of $\boldsymbol{\sigma}$, and

$$\sigma'_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{1.4.8}$$

is the deviatoric part of $\boldsymbol{\sigma}$. Furthermore, the following statements can be easily proven:

1. $\mathbf{A} = 3a\mathbf{I}^h + 2b\mathbf{I}^d$ is positive definite if and only if $a > 0$, $b > 0$.
2. $\mathbf{A}^{-1} = \dfrac{1}{3a}\mathbf{I}^h + \dfrac{1}{2b}\mathbf{I}^d$.
3. $\mathbf{I} = \mathbf{I}^h + \mathbf{I}^d$ is the identity tensor, that is, $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A}$.
4. If $\mathbf{B} = 3c\mathbf{I}^h + 2d\mathbf{I}^d$, then $\mathbf{A} + \mathbf{B} = 3(a + c)\mathbf{I}^h + 2(b + d)\mathbf{I}^d$.
5. $\mathbf{AB} = \mathbf{BA} = 9ac\mathbf{I}^h + 4bd\mathbf{I}^d$.

Additionally, we introduce another symbolic notation for fourth-order isotropic tensors:

$$\mathbf{A} = (3a, 2b). \tag{1.4.9}$$

The following statements can be easily proven:

1. $\mathbf{A} = (3a, 2b)$ is positive definite if and only if $a > 0$, $b > 0$.
2. $\mathbf{A}^{-1} = \left(\dfrac{1}{3a}, \dfrac{1}{2b}\right)$.
3. $\mathbf{I} = (1,1)$ is the identity tensor, that is, $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = (1,1)$.
4. If $\mathbf{B} = (3c, 2d)$, then $\mathbf{A} + \mathbf{B} = (3a + 3c, 2b + 2d)$.
5. $\mathbf{AB} = \mathbf{BA} = (9ac, 4bd)$.

For tensor calculus, the most often used operations are the gradient and divergence. The gradient of a scalar function results in a first-order tensor:

$$\mathbf{b} = \nabla f \Leftrightarrow b_j = \frac{\partial f}{\partial x_j} \equiv f_{,j}. \tag{1.4.10}$$

The divergence of a vector is a scalar:

$$a = \text{div}[\mathbf{b}] = \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3} \Leftrightarrow a = b_{j,j}. \tag{1.4.11}$$

In the above, the notation $(\bullet)_{,j}$ indicates the derivative with respect to the independent variable of the function. If the function has more than one independent variable, ambiguity may arise. In this case, instead of using, for example, $f_{,j}(x, y)$, we will use the standard notation to explicitly indicate partial derivatives, for example,

$$\frac{\partial f(x, y)}{\partial x}. \tag{1.4.12}$$

The divergence of a second-order tensor becomes a vector:

$$\mathbf{p} = \nabla \cdot \boldsymbol{\sigma} \Leftrightarrow p_i = \frac{\partial \sigma_{ij}}{\partial x_j} = \sigma_{ij,j}. \tag{1.4.13}$$

For integrals, a single integral sign will be used. For line (one-dimensional) integrals, the integration variable will be used for the infinitesimal line element, for example,

$$y = \int_L f(x)\, dx,$$

where L is the line of integration. For surface (two-dimensional) integrals, we will typically use dS for the infinitesimal area element, for example,

$$y = \int_S f(\mathbf{x})\, dS,$$

where S is the area of integration. Here, it is implied that $\mathbf{x}$ is the integration variable since the integrand depends on $\mathbf{x}$ only. If the integrand depends on more than one independent variable, we will explicitly indicate which variable is being integrated, for example,

$$y(\mathbf{z}) = \int_S f(\mathbf{x}, \mathbf{z})\, dS(\mathbf{x}).$$

Similarly, for volume (three-dimensional) integrals, we will use either one of the following two forms:

$$y = \int_V f(\mathbf{x})\, dV, \qquad y(\mathbf{z}) = \int_V f(\mathbf{x}, \mathbf{z})\, dV(\mathbf{x}).$$

Finally, we introduce some special tensors. The Kronecker delta δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}. \tag{1.4.14}$$

And the permutation tensor ε_{ijk} is defined by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{when } i, j, k \text{ are an even permutation of } 1, 2, 3 \\ -1 & \text{when } i, j, k \text{ are an odd permutation of } 1, 2, 3 \\ 0 & \text{when any two indices are equal} \end{cases}. \tag{1.4.15}$$

For example, $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} = -1$, and others are zero.

The permutation tensor and the Kronecker delta δ_{ij} are related through the $\varepsilon - \delta$ relationship:

$$\varepsilon_{ijk}\varepsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}. \tag{1.4.16}$$

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