# 1

# INTRODUCTION

> "Reeling and Writhing, of course, to begin with," the Mock Turtle replied,
> "and the different branches of Arithmetic—
> Ambition, Distraction, Uglification and Derision."
>
> —Lewis Carroll

The word *algorithm* is central to mathematics and computer science. It means a step-by-step problem solving procedure. All of us have learned many algorithms in our schooling. The algorithm for "long division" is a good example. I still recall the oversimplified version we students pronounced as a refrain for our fourth-grade teacher: "divide, multiply, subtract, bring down, repeat." It never occurred to that teacher—or to us students, of course—that it might be appropriate for us to understand what was going on as we carried out what were to us a series of mindless rote activities.

We should always have been concerned with why algorithms like that work. Fortunately, the computer revolution has brought that concern to the forefront.

The seeds of my own investigation of algorithms were planted at a math meeting in Kansas City in 1972. A Hewlett-Packard Company representative who had attended a talk I gave on computation caught my attention and introduced himself. At the time that company was producing some of the early desktop electronic calculators, so I was pleased to talk with him.

*Inside Your Calculator: From Simple Programs to Significant Insights* By Gerald R. Rising

**Figure 1.1** The HP-35 scientific calculator of the 1970s.

"I've got something to show you," he told me slyly, the twinkle in his eye reminding me of those grifters whose inside pockets are filled with stolen wristwatches. He did not, however, have contraband for sale. Instead he took from his jacket pocket a small leather-encased parcel. He opened it to disclose the first handheld scientific calculator, an HP-35 (Figure 1.1).

As I write this over 30 years later, I find it difficult to communicate the astonishment I felt on that morning. Until that time the only electronic calculators available were "four-bangers"—so called because their processing was limited to the four fundamental operations of paper-and-pencil arithmetic: addition, subtraction, multiplication, and division. Even those had not been around for long. The first handheld calculators had been available only since 1970, the first electronic desktop calculators since 1963. Here was a calculator that not only performed those four operations but also—for the first time—calculated trigonometric and logarithmic functions, reciprocals, and roots.

Still more impressive to me, when I punched the appropriate keys to enter $2.356^{3.71}$, the calculator almost instantaneously displayed 24.03091523.

To gain some sense of both my astonishment and the remarkable power this tiny instrument provided, consider how I would have had to address that problem at that time. (Logarithms will be reviewed later, and you do not have to follow the details of this worked example to understand my point.)

I would have written the exercise as an equation $x = 2.356^{3.71}$, then taken the logarithm of each side, in the process applying one of the rules of logs to the right side:[1] $\log x = 3.71 * \log 2.356$.

Next I would have looked up log 2.356 in a (base 10) log table, interpolating[2] to give .3722, annexed the appropriate characteristic, 0, and substituted it in that equation: $\log x = 3.71 * 0.3722$.

Now I would have multiplied those right-side factors with a simple calculator[3] to produce $\log x = 1.3809$.

Finally, I would have returned to log tables to find, by interpolating again, the antilog of 1.3809 to arrive finally at $x = 24.04$.

Notice several things about that processing. First, of course, it was lengthy and time-consuming—and I have not even included the interpolation procedures. Moreover, it gave nowhere near the number of decimal places of the calculator answer; and finally, the answer it did produce was not even accurate to that fourth digit.[4]

Knowing all this, I was stunned. A single exercise that would have taken me at least 5 minutes was now calculated as fast as I could key in the numbers and operation. Electronic engineers had been able to pack into this tiny device tremendous computing power, and I could not imagine how they performed this feat of calculating wizardry.

It turns out, I now learn, that even the manufacture of this calculator was a kind of fluke. When one of the early electronic desktop calculators was developed that would compute with this power, William Hewlett, the head of Hewlett-Packard Company, was impressed with the small space taken up inside the case by its electronic components. He asked his engineers if they could squeeze this power into a shirtpocket-sized calculator. The engineers responded with the HP-35. At first they planned to make only a few: for their boss, other company administrators, their engineering colleagues, and, of course, themselves. Fortunately for

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[1] A reminder: throughout this book, * represents multiplication.

[2] Sadly, one of the "benefits" for calculators that has attracted some teachers to them is that their extra digits "eliminate the need for interpolation." See Appendix B for more on this important mathematical technique.

[3] Without an electronic calculator that processed arithmetic, I would have had to choose between multiplying those numbers by paper and pencil or taking logs again, complicating the calculation still further.

[4] Some older readers may recall that a slide rule would have simplified matters for those who knew how to use one, but even less accuracy would have been possible: to three digits on all but a few expensive models that provided four.

them they changed their minds because, by the time this calculator was finally outmoded a few years later, tens of thousands had been sold—for $395 each!

Of course, within a few years Japanese manufacturers would flood the market with inexpensive four-bangers and scientific calculators. At the height of those times when—as what was called Japan, Inc.—our defeated World War II opponent seemed about to reverse the results of that war economically, calculator prices reached rock bottom. Even programmable calculators could then be purchased for under $10.

But at that meeting my imagination was captured by that tiny instrument. I wanted to know what was happening inside that calculator. How did it magically produce those results? How did those engineers accomplish this further advance in computation?

I knew that electronic engineers had several things going for them that their predecessors did not have. Their electronics gave them:

1. Great computational speed
2. Large storage (memory) capacity
3. Programming opportunities

Somehow they had harnessed those electronic gifts to produce what were to me and others such brilliant results.

## THE BLACK BOX

The problem I faced, it seemed to me, was related to a pedagogical device that is useful for encouraging students at almost any level to think seriously about mathematics. The teaching device is often called a *black box* or *function machine*, and the challenge is *What's My Rule?*

The students are presented with an imaginary black box into which you can feed a number. Each time you enter a number you receive in response a corresponding answer number. By testing with as many input numbers as you wish, you are asked to determine what is happening inside the black box, what mathematical operations are doing to that input number to manufacture that output number.

The students don't even have to state the rule in words. By showing their teacher that they can correctly predict what output number results from the input numbers with which they are challenged, they demonstrate that they know "what is going on in the box." (This way they also don't disclose the secret to others who can continue to seek the rule.)

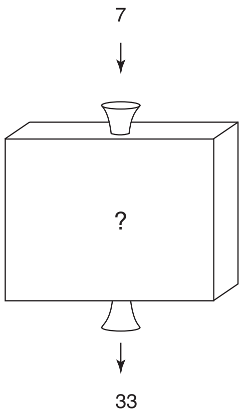

**Figure 1.2** A function “machine.”

Consider a sample black box problem. You are to determine what the following box does to numbers dropped in the top funnel. Shown in Figure 1.2 is a 7 entered to produce 33.[5]

By trying other input values, students would seek to determine the rule that would produce these results. Mathematicians recognize such boxes as the equivalent of functions.

I, too, sought to determine what is going on in a black box, except that my box was equivalent to a scientific calculator key. Also my search was different from the search in *What’s My Rule?* I know the rule; it is stamped on or near that key. What I sought and what this book is about is how that rule might be accomplished—for example, what could be going on inside that black box labeled $\boxed{\text{COS}}$ (see Figure 1.3).

In this task trial and error will not suffice, however. You can enter value after value to obtain outcome after outcome without making much progress in determining what is going on inside the box. So a quite different approach is required. In order to answer this question you have to explore the mathematics of the function cosine[6] as well as the programming necessary to support that math. And that is what you will do in the remainder of this book.

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[5] Without further information, there is a wide range of possibilities for this box rule. If we consider the input value as $x$, the rule could be $5x - 2$ or $x^2 - 16$ or even just 33 for every input $x$. Mathematicians know that the rule for a finite number of specific input values need not be unique, but that does not affect the game as it is played with less sophisticated contestants.

[6] I have arbitrarily chosen cosine, abbreviated $\boxed{\text{COS}}$ on the calculator, to represent one of the circular or trigonometric functions. Once we have the means for calculating its values, the sine, $\boxed{\text{SIN}}$, and tangent, $\boxed{\text{TAN}}$, keys are quickly determined by use of trigonometric identities.

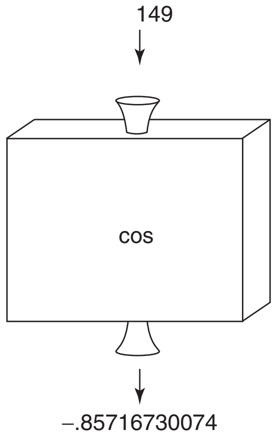

**Figure 1.3** The $\boxed{\text{COS}}$ key as a black box.

## SUPERHUMAN ENGINEERS?

I have always held engineers in very high regard. They are the "Can do!" people of this world. Given a practical problem, they set out to solve it. Build a dam, erect a skyscraper, construct a road, send a rocket to the moon: they get at it. I honor them for their creativity and their work ethic.[7] But at first the awe I felt for engineers got in the way of my figuring out how computers calculate. I was certain that they were applying some very advanced and highly abstruse math in extraordinarily complex programs to solve these problems.

That it turned out otherwise came as a revelation to me as I hope it will to you. In the following chapters you will meet simple programs that carry out the functions of those calculator keys. In the process you should gain further insights into the mathematical and programming concepts that support them, insights that should serve you well in other contexts. And the number of program steps needed to carry out these tasks is several orders of magnitude fewer than those that support contemporary computer games.

Here, for example, is a seven-line program that will calculate the cosine for the input of any number of degrees to nine- or ten-digit accuracy:[8]

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[7] Basing my judgment on my over 40 years of university lecturing, I prefer teaching classes of engineers to all other students.

[8] Those who wish to enter this program should find Appendix A on programming specifics useful.

```
PROGRAM:COSDEG
: Prompt X
: X*π/180→X
: X*X/4294967296→S
: For (I,1,16)
:    S(4−S)→S
: End (For)
: Disp 1−S/2
```

If you enter those seven program lines in a programmable calculator and run the program, it will carry out this seemingly formidable task.

Suppose now, for example, that you wish to calculate the cosine of 149°. When you run the program, the calculator will display `X=?`, to which you would respond by keying 149 and pressing $\boxed{\text{ENTER}}$.[9] Your display would then look like this:

```
X=?149
    −.8571673008
            Done
```

You will see how and why that remarkable program works in Chapter 8. For now I want only to show you the tasks those steps are performing:

`Prompt X` This is the program line that displays that `X?` when the program is run, inviting you to type in a number of degrees.

`X*π/180→X` In this line the number you entered in response to the prompt in line 1 is multiplied by $\pi$. The value of $\pi$ (3.1415926535898) is stored[10] in calculator memory. The product you attain is then divided by 180. The arrow tells you to store this result in `X`, replacing any value that was there previously.

`X*X/4294967296→S` More arithmetic. Our new stored `X` is multiplied by itself and then divided by that strange number, which looks

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[9] Although it is not necessary in order to follow the arguments in this book, I strongly encourage you to run this and other programs. To run this program it is not necessary to set your calculator in `Degree` mode, but to check it you would need to do so.

[10] Most calculators will show stored values like $\pi$ to the limited number of digits of the display, but will carry more digits in memory. To check any calculator to determine how many digits of $\pi$ are stored, enter $\pi$, subtract 3.1415926, and multiply by 10000000. If your calculator value for $\pi$ had been the 3.141592654 displayed, the result of this calculation would have been .54. When you see something different from this, in the case of the TI-84 .535898, those digits replace the 54 to give us 3.1415926535898. Thus this calculator carries $\pi$ accurate to 14 digits. Other calculators and computers will differ, and it is an interesting task to check them out, not just for $\pi$ but for calculated values such as $\sqrt{2}$ as well.

like the national debt and happens to be $2^{32}$. The quotient, clearly a very small number, is stored in `S`.

```
For (I,1,16)
S(4–S)→S
```

`End` These three lines form what is called a *counting loop*. In the `For` line an internal counter `I` runs the lines between it and the `End` line 16 times. (It "counts" from the first value, 1, to 16.) Of course, here there is only one line to be calculated over and over. It takes the current value of `S`, multiplies it by 4 minus that same value, and stores the result back in `S`. (You can think of the `For` statement as "For I taking values from 1 to 16, do the following:.")

The power of this control stucture is displayed clearly here. These three lines replace 16 program lines, all alike:

```
S(4-S)→S
S(4-S)→S
...
S(4-S)→S
```

`Disp 1–S/2` One last minor bit of arithmetic. The final `S` result produced by that `For` loop is divided by 2 and the result subtracted from 1. The answer is then displayed. It is the cosine of your original input `X`, in our example 149°.

To understand what went on in those steps, readers not already familiar with simple programming will have had to learn from this analysis about input `(Prompt)`, output `(Disp)`, and calculator storage `(→)` and how a particular loop `(For)` works. Aside from those programming features, however, all that is involved here is simple arithmetic, in this case subtraction, multiplication, and division. Of course, that means simple for the calculator! None of us would want to divide by 4294967296 or carry out even one of those 16 multiplications of 10-digit factors without access to such a device.

Before closing this discussion, I must enter an important reservation. Some of you will have checked the result of our program calculation of cosine 149°. Our result, −.8571673008, does not quite agree with the result you obtain when you simply use the scientific calculator keyboard to key $\boxed{\text{COS}}$ 149 and press $\boxed{\text{ENTER}}$. If you do that, the calculator will display `−.8571673007`,[11] which differs by one in that tenth place from

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[11] If you do that and get the wildly different answer, −.2237409501, your calculator is in `Radian` mode and you must change it to `Degree` mode.

our program calculation. An error of that magnitude corresponds to a measurement that is off by less than an inch in 100,000 miles, but the two values do differ, suggesting that they are arrived at by different internal processing avenues.

In fact, cosine is calculated by most computers and calculators by a quite different (and still faster) program. You will meet a simulation of that program in Chapter 9—not, however, to make this tiny correction but only because it involves additional interesting mathematics and programming.

## JUST HOW POWERFUL IS THAT PROGRAM?

I invite you to examine the power of that seven-line program by comparing it with what you would have needed to do to accomplish the same result before the advent of electronic calculation.[12]

Consider Table 1.1, which gives only the values for angles in whole-number degrees. If you wished to give values for tenths of a degree ($1.0^\circ$, $1.1^\circ$, $1.2^\circ$, etc.), the table would have to be 10 times as long. Similarly, for values to hundredths of a degree ($1.01^\circ$, $1.02^\circ$, $1.03^\circ$, etc.), it would have to be 100 times as long.

But our seven-line program provides values for angles measured to hundred millionths of a degree. For example, our scientific calculator tells us that $\cos 45.12345678^\circ = .0705581518$. To provide all this information (even allowing for linear interpolation), a table 10,000,000 times as long as Table 1.1 would be necessary. Whereas Table 1.1, in degrees, takes up one page, a table that would provide all this information would call for 10 million pages. That is, of course, a great many pages. It would take twenty thousand 500-page volumes to include all of them, an entire library devoted to the values corresponding to this one of the many calculations that our little calculators so simply programmed can perform.

Two reasonable questions arise at this point. First, angles are rarely given in decimal values. Instead, like hours, degrees are broken down into 60 minutes and each minute into 60 seconds. For example, you might have an angle of $27^\circ 39' 12''$, that notation representing 27 degrees, 39 minutes, and 12 seconds. While some calculators allow input in this form, many do not. It is reasonably simple to convert from one form to the other,

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[12] This discussion is oversimplified. More information could be included on each page, and various shortcuts are employed by books of tables to reduce the number of pages. However, the message remains: a remarkable amount of information may be retrieved through calculator processing.

**TABLE 1.1. Cosine Values from 0° to 90°**

| Degrees | Cosine | Degrees | Cosine |
|---|---|---|---|
| 0 | 1.000000000 | 45 | .707106781 |
| 1 | .999847695 | 46 | .694658370 |
| 2 | .999390827 | 47 | .681998360 |
| 3 | .998629535 | 48 | .669130606 |
| 4 | .997564050 | 49 | .656059029 |
| 5 | .996194698 | 50 | .642787610 |
| 6 | .994521895 | 51 | .629320391 |
| 7 | .992546152 | 52 | .615661475 |
| 8 | .990268069 | 53 | .601815023 |
| 9 | .987688341 | 54 | .587785252 |
| 10 | .984807753 | 55 | .573576436 |
| 11 | .981627183 | 56 | .559192903 |
| 12 | .978147601 | 57 | .544639035 |
| 13 | .974370065 | 58 | .529919264 |
| 14 | .970295726 | 59 | .515038075 |
| 15 | .965925826 | 60 | .500000000 |
| 16 | .961261696 | 61 | .484809620 |
| 17 | .956304756 | 62 | .469471563 |
| 18 | .951056516 | 63 | .453990500 |
| 19 | .945518576 | 64 | .438371147 |
| 20 | .939692621 | 65 | .422618262 |
| 21 | .933580426 | 66 | .406736643 |
| 22 | .927183855 | 67 | .390731128 |
| 23 | .920504853 | 68 | .374606593 |
| 24 | .913545458 | 69 | .358367950 |
| 25 | .906307787 | 70 | .342020143 |
| 26 | .898794046 | 71 | .325568154 |
| 27 | .891006524 | 72 | .309016994 |
| 28 | .882947593 | 73 | .292371705 |
| 29 | .874619707 | 74 | .275637356 |
| 30 | .866025404 | 75 | .258819045 |
| 31 | .857167301 | 76 | .241921896 |
| 32 | .848048096 | 77 | .224951054 |
| 33 | .838670568 | 78 | .207911691 |
| 34 | .829037573 | 79 | .190808995 |
| 35 | .819152044 | 80 | .173648178 |
| 36 | .809016994 | 81 | .156434465 |
| 37 | .798635510 | 82 | .139173101 |
| 38 | .788010754 | 83 | .121869343 |
| 39 | .777145961 | 84 | .104528463 |
| 40 | .766044443 | 85 | .087155743 |
| 41 | .754709580 | 86 | .069756474 |
| 42 | .743144825 | 87 | .052335956 |
| 43 | .731353702 | 88 | .034899497 |
| 44 | .719339800 | 89 | .017452406 |
| 45 | .707106781 | 90 | .000000000 |

however. There are 60 minutes in a degree and 60 times 60 seconds (thus 3600 seconds) in a degree, so you calculate

$$27^\circ 39' 12'' = \left(27 + \frac{39}{60} + \frac{12}{3600}\right)^\circ = 27.65333333^\circ$$

and find the cosine of this result. This gives $\cos 27^\circ 39' 12'' = .8857719399$.

The second question deserves a serious response. Is all this accuracy of any real value? Cannot you get along just as well with the four- or five-digit accuracy of those older days?

Mathematics, science, and engineering teachers at all levels should be especially sensitive to such questions for they face the obverse of this problem: their students are quite content with meaningless accuracy. When asked to find the circumference of a circle with a 12.0 inch diameter,[13] for example, those youngsters who know the formula $C = \pi d$ will enter 12 into a calculator with 10-digit display and multiply it by the calculator value of $\pi$ to obtain 37.69911184. They are quite satisfied then to give 37.69911184 inches as their answer when most of those decimal digits unnecessarily confound the problem. A more appropriate answer would be 37.7 inches.[14]

But we are still faced with this fair inquiry: Is all this extra accuracy ever of any value?

An example should respond to this question. Global Positioning System satellite (GPS) devices (e.g., see Figure 1.4) are widely used today. GPS instruments, formerly equipment restricted to the armed forces, have become widely available. They are used by surveyors; travelers by airplane, car, and boat; hunters and explorers; and parcel delivery personnel. Although these tools, the size of handheld calculators, provide many other features—most notably maps—their basic function is to determine your location on the earth; that is, your latitude and longitude. This is accomplished by calculating your position in relation to a number of satellites.

When I take my GPS out in my backyard and turn it on, it reports my latitude as $44^\circ 00.179'$ north, my longitude as $78^\circ 44.932'$ west. It also reports how accurate these values are, this accuracy depending on the

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[13] The measure 12.0 inches differs from the measure 12 inches. The measure 12.0 inches indicates that the measure is to the nearest tenth of an inch; the measure 12, to the nearest inch. In formal terms, if a measurement $m$ is 12.0, then $11.95 \leq m < 12.05$. If, on the other hand, a measurement $n$ is 12, then $11.5 \leq n < 12.5$.

[14] There are rules governing significant digits obtained from calculations with numbers arrived at through measurement. Note that in this case we have simply retained the same number of digits—three—as the given diameter 12.0. It should also be noted that there is an exact, but abstract and not useful for measurement, answer to this question: $12\pi$.

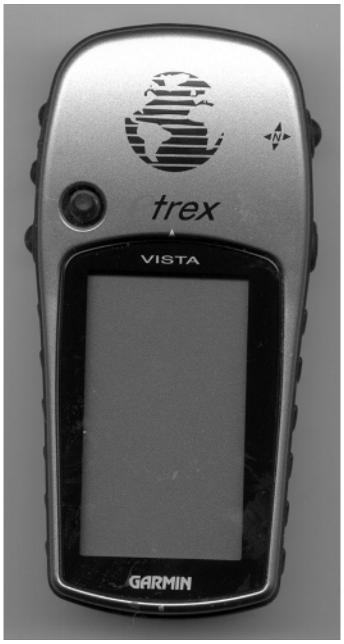

**Figure 1.4** A GPS device.

number of satellites it can "see" from this location.[15] When I took that reading, for example, my GPS reported "Accurate to 19 feet."

Even readers unfamiliar with GPS devices have almost surely seen their contribution under less happy circumstances. Those laser-guided "smart" bombs employed by armed forces that miss their targets by at most a few feet have similar tools built into them. World War II veterans like me are especially impressed by that accuracy. During that war just 60 years ago, the proportion of bombs dropped by our B-17 Flying Fortresses over Europe that fell within 1000 feet of a designated target was only 20%. Those were daylight raids; the British nighttime bombing was still less accurate.[16]

If those examples don't make the case for that many digits, you need only extend these kinds of tasks to the accurate location of spacecraft as they travel through the solar system and beyond. An idea of the kind

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[15] Under a heavy canopy of trees, GPS devices are of little use. Thus open areas and winter provide better accuracy.

[16] This information is derived from page 5 of Franklin D'Olier et al., *The United States Strategic Bombing Survey Summary Report* (European War), available from `www.anesi.com/ussbs02.htm`.

of accuracy called for in astronomical measurements is suggested by the definition of the meter to be the length of a path traveled by light through a vacuum in .000000003335640952 second.

Where does the remarkable precision of these contemporary devices come from? It should be evident that one source of this precision is our ability to calculate to extreme accuracy.

## WHAT LIES AHEAD

In the remainder of this book I will share with you the results of my exploration of electronic calculation. Included will be the basis for that cosine program and the other features that make the scientific calculator so different from the four-bangers that came before them.

But first we must set the stage for modern electronic calculation. In Chapter 2 you will meet some additional mathematical background to support this history. Then you will examine some remarkable algorithms that could be used to back up the calculator keys $\boxed{\sqrt{\ }}$, $\boxed{\text{cos}}$, $\boxed{\text{log}}$ and $\boxed{x^y}$. (General readers may wish to review the history preceding electronic calculation in Appendix C before continuing with Chapter 2.)