Chapter 4

EQUATIONS, RATIOS, AND PROPORTIONS

Chapter Check-In

- ❑ Axioms of equality
- ❑ Solving equations
- ❑ Solving proportions for value

Working with variables and solving equations are often considered the basis of algebra.

Equations

An **equation** is a mathematical sentence, a relationship between numbers and/or symbols.

Axioms of equality

For all real numbers *a, b,* and *c,* the following are some basic rules for using the equal sign.

- **Reflexive axiom:** $a = a$.

  Therefore, $4 = 4$.

- **Symmetric axiom:** If $a = b$, then $b = a$.

  Therefore, if $2 + 3 = 5$, then $5 = 2 + 3$.

- **Transitive axiom:** If $a = b$ and $b = c$, then $a = c$.

  Therefore, if $1 + 3 = 4$ and $4 = 2 + 2$, then $1 + 3 = 2 + 2$.

- **Additive axiom:** If $a = b$ and $c = d$, then $a + c = b + d$.

  Therefore, if $1 + 1 = 2$ and $3 + 3 = 6$,
  then $1 + 1 + 3 + 3 = 2 + 6$.

- **Multiplicative axiom:** If $a = b$ and $c = d$, then $ac = bd$.

  Therefore, if $1 = 2/2$ and $4 = 8/2$, then $1(4) = (2/2)(8/2)$

Solving equations

Remember that an equation is like a balance scale with the equal sign (=) being the fulcrum, or center. Thus, if you do the *same thing to both sides* of the equal sign (say, add 5 to each side), the equation will still be balanced.

Example 1: Solve for *x*.

$$x - 5 = 23$$

To solve the equation $x - 5 = 23$, you must get x by itself on one side; therefore, add 5 to both sides.

$$\begin{array}{r} x - 5 = 23 \\ +5 \quad +5 \\ \hline x \qquad = 28 \end{array}$$

In the same manner, you may subtract, multiply, or divide *both* sides of an equation by the same (nonzero) number, and the equation will not change. Sometimes you may have to use more than one step to solve for an unknown.

Example 2: Solve for *x*.

$$3x + 4 = 19$$

Subtract 4 from both sides to get the $3x$ by itself on one side

$$\begin{array}{r} 3x + 4 = 19 \\ -4 \quad -4 \\ \hline 3x \qquad = 15 \end{array}$$

Then divide both sides by 3 to get *x*.

$$\frac{3x}{3} = \frac{15}{3}$$
$$x = 5$$

Remember that solving an equation is using opposite operations until the letter is on a side by itself (for addition, subtract; for multiplication, divide, and so forth).

To check, substitute your answer into the original equation.

$$3x + 4 = 19$$
$$3(5) + 4 = 19$$
$$15 + 4 = 19$$
$$19 \stackrel{\checkmark}{=} 19$$

Example 3: Solve for *x*.

$$\frac{x}{5} - 4 = 2$$

Add 4 to both sides.

$$\begin{array}{rcr} \frac{x}{5} - 4 & = & 2 \\ +4 & & +4 \\ \hline \frac{x}{5} & = & 6 \end{array}$$

Multiply both sides by 5 to get *x*.

$$(5)\frac{x}{5} = (5)6$$
$$x = 30$$

Example 4: Solve for *x*.

$$\frac{3}{5}x - 6 = 12$$

Add 6 to each side.

$$\begin{array}{r} \frac{3}{5}x - 6 = 12 \\ +6 \quad +6 \\ \hline \frac{3}{5}x \quad = 18 \end{array}$$

Multiply each side by 5/3 (same as dividing by 3/5).

$$\left(\frac{5}{3}\right)\frac{3}{5}x = \left(\frac{5}{3}\right)18$$

$$x = \left(\frac{5}{\cancel{3}_1}\right)\frac{\cancel{18}^6}{1}$$

$$x = 30$$

Example 5: Solve for *x*.

$$5x = 2x - 6$$

Add –2*x* to each side.

$$\begin{array}{r} 5x = \quad 2x - 6 \\ -2x \quad -2x \quad \\ \hline 3x = \quad\quad -6 \end{array}$$

Divide both sides by 3.

$$\frac{3x}{3} = \frac{-6}{3}$$

$$x = -2$$

Example 6: Solve for *x*.

$$6x + 3 = 4x + 5$$

Add –3 to each side.

$$\begin{array}{r} 6x + 3 = 4x + 5 \\ -3 \quad\quad -3 \\ \hline 6x \quad = 4x + 2 \end{array}$$

Add $-4x$ to each side.

$$\begin{array}{rcr} 6x & = & 4x+2 \\ -4x & & -4x \\ \hline 2x & = & 2 \end{array}$$

Divide each side by 2.

$$\frac{2x}{2} = \frac{2}{2}$$
$$x = 1$$

Literal equations

Literal equations have no numbers, only symbols (letters).

Example 7: Solve for Q.

$$QP - X = Y$$

First add X to both sides.

$$\begin{array}{rcl} QP - X & = & Y \\ +X & & +X \\ \hline QP & = & Y + X \end{array}$$

Then divide both sides by P.

$$\frac{QP}{P} = \frac{Y+X}{P}$$
$$Q = \frac{Y+X}{P}$$

Operations opposite to those in the original equation were used to isolate Q. (To remove the $-X$, a $+X$ was *added* to both sides of the equation. Because the problem has Q times P, both sides were *divided* by P.

Example 8: Solve for y.

$$\frac{y}{x} = c$$

Multiply both sides by x to get y alone.

$$(x)\frac{y}{x} = (x)c$$
$$y = xc$$

Example 9: Solve for x.

$$\frac{b}{x} = \frac{p}{q}$$

To solve this equation quickly, you cross multiply. To cross multiply,

1. **Bring the denominators up next to the opposite side numerators and**
2. **Multiply**

$$\frac{b}{x} = \frac{p}{q}$$
$$bq = px$$

Then divide both sides by p to get x alone.

$$\frac{bq}{p} = \frac{px}{p}$$
$$\frac{bq}{p} = x \text{ or } x = \frac{bq}{p}$$

Cross multiplying can be used only when the format is two fractions separated by an equal sign.

Be aware that cross multiplying is most effective only when the letter you are solving for is on the *bottom* (the denominator) of a fraction. If it is on top (the numerator), it is easier simply to clear denominator under the unknown you're solving for.

Example 10: Solve for x.

$$\frac{x}{k} = \frac{p}{q}$$

Multiply both sides by *k*.

$$(k)\frac{x}{k} = (k)\frac{p}{q}$$

$$x = \frac{kp}{q}$$

In this problem, there is no need to cross multiply.

Ratios and Proportions

Ratios and proportions are not only used in arithmetic, but are also commonly used in algebra (and geometry). The definitions given in this chapters are the same as those used in arithmetic.

Ratios

A **ratio** is a method of comparing two or more numbers or variables. Ratios are written as *a:b* or in working form, as a fraction.

$$a/b \text{ or } \frac{a}{b}$$

is read "*a* is to *b*." Notice that whatever comes after the "to" goes second or at the bottom of the fraction.

Proportions

Proportions are written as two ratios (fractions) equal to each other.

Example 11: Solve this problem for *x*.

$$p \text{ is to } q \text{ as } x \text{ is to } y$$

First the proportion may be rewritten.

$$\frac{p}{q} = \frac{x}{y}$$

Now simply multiply each side by *y*.

$$(y)\frac{p}{q} = (y)\frac{x}{y}$$

$$\frac{yp}{q} = x$$

Example 12: Solve this proportion for *t*.

s is to t as r is to q

Rewrite.

$$\frac{s}{t} = \frac{r}{q}$$

Cross multiply.

$$sq = rt$$

Divide both sides by *r*.

$$\frac{sq}{r} = \frac{rt}{r}$$

$$\frac{sq}{r} = t$$

Solving proportions for value

Follow the procedures given in Examples 11 and 12 to solve for the unknown.

Example 13: Solve for *x*.

$$\frac{4}{x} = \frac{2}{5}$$

Cross multiply.

$$(4)(5) = 2x$$

$$20 = 2x$$

Divide both sides by 2.

$$\frac{20}{2} = \frac{2x}{2}$$

$$10 = x$$

Chapter Checkout

Q&A

1. True or false: If $a = b$ and $b = c$, then $a = c$.
2. Solve for x: $\frac{x}{4} - 5 = 8$
3. Solve for x: $7x + 3 = 5x + 7$
4. Solve for m: $mn - r = q$
5. Solve for x: $\frac{a}{x} = \frac{b}{c}$
6. Solve for y: m is to n as y is to z.
7. Solve for x: $\frac{6}{x} = \frac{3}{5}$

Answers: 1. True **2.** 52 **3.** 2 **4.** $\frac{q + r}{n}$ **5.** $\frac{ac}{b}$ **6.** $\frac{mz}{n}$ **7.** 10