PART One

Quantitative Analysis

CHAPTER 1

Bond Fundamentals

Risk management starts with the pricing of assets. The simplest assets to study are regular, fixed-coupon bonds. Because their cash flows are predetermined, we can translate their stream of cash flows into a present value by discounting at a fixed interest rate. Thus, the valuation of bonds involves understanding compounded interest, discounting, as well as the relationship between present values and interest rates.

Risk management goes one step further than pricing, however. It examines potential changes in the price of assets as the interest rate changes. In this chapter, we assume that there is a single interest rate, or yield, that is used to price the bond. This will be our fundamental risk factor. This chapter describes the relationship between bond prices and yields and presents indispensable tools for the management of fixed-income portfolios.

This chapter starts our coverage of quantitative analysis by discussing bond fundamentals. Section 1.1 reviews the concepts of discounting, present values, and future values. Section 1.2 then plunges into the price-yield relationship. It shows how the Taylor expansion rule can be used to relate movements in bond prices to those in yields. This Taylor expansion rule, however, covers much more than bonds. It is a building block of risk measurement methods based on local valuation, as we shall see later. Section 1.3 then presents an economic interpretation of duration and convexity.

The reader should be forewarned that this chapter, like many others in this handbook, is rather compact. This chapter provides a quick review of bond fundamentals, with particular attention to risk measurement applications. By the end of this chapter, however, the reader should be able to answer advanced FRM questions on bond mathematics.

1.1 DISCOUNTING, PRESENT, AND FUTURE VALUE

An investor considers a zero-coupon bond that pays $100 in 10 years. Assume that the investment is guaranteed by the U.S. government, and that there is no credit risk. So, this is a default-free bond, which is exposed to market risk only. Because the payment occurs at a future date, the current value of the investment is surely less than an up-front payment of $100.

To value the payment, we need a **discounting factor**. This is also the **interest rate**, or more simply, the **yield**. Define C_t as the cash flow at time t and the discounting factor as y. We define T as the number of periods until maturity, such as a number of years, also known as **tenor**. The **present value** (PV) of the bond can be computed as

$$PV = \frac{C_T}{(1+y)^T} \tag{1.1}$$

For instance, a payment of $C_T = \$100$ in 10 years discounted at 6% is only worth \$55.84 now. So, all else fixed, the market value of zero-coupon bonds decreases with longer maturities. Also, keeping T fixed, the value of the bond decreases as the yield increases.

Conversely, we can compute the **future value** (FV) of the bond as

$$FV = PV \times (1+y)^T \tag{1.2}$$

For instance, an investment now worth $PV = \$100$ growing at 6% will have a future value of $FV = \$179.08$ in 10 years.

Here, the yield has a useful interpretation, which is that of an **internal rate of return** on the bond, or annual growth rate. It is easier to deal with rates of returns than with dollar values. Rates of return, when expressed in percentage terms and on an annual basis, are directly comparable across assets. An annualized yield is sometimes defined as the **effective annual rate (EAR)**.

It is important to note that the interest rate should be stated along with the method used for compounding. Annual compounding is very common. Other conventions exist, however. For instance, the U.S. Treasury market uses semiannual compounding. Define in this case y^S as the rate based on semiannual compounding. To maintain comparability, it is expressed in annualized form, i.e., after multiplication by 2. The number of periods, or semesters, is now $2T$. The formula for finding y^S is

$$PV = \frac{C_T}{(1+y^S/2)^{2T}} \tag{1.3}$$

For instance, a Treasury zero-coupon bond with a maturity of $T = 10$ years would have $2T = 20$ semiannual compounding periods. Comparing with (1.1), we see that

$$(1+y) = (1+y^S/2)^2 \tag{1.4}$$

Continuous compounding is often used when modeling derivatives. It is the limit of the case where the number of compounding periods per year increases to infinity. The continuously compounded interest rate y^C is derived from

$$PV = C_T \times e^{-y^C T} \tag{1.5}$$

where $e^{(\cdot)}$, sometimes noted as $\exp(\cdot)$, represents the exponential function.

Note that in Equations (1.1), (1.3), and (1.5), the present value and future cash flows are identical. Because of different compounding periods, however, the yields will differ. Hence, the compounding period should always be stated.

Example: Using Different Discounting Methods

Consider a bond that pays \$100 in 10 years and has a present value of \$55.8395. This corresponds to an annually compounded rate of 6.00% using $PV = C_T/(1+y)^{10}$, or $(1+y) = (C_T/PV)^{1/10}$.

This rate can be transformed into a semiannual compounded rate, using $(1+y^S/2)^2 = (1+y)$, or $y^S/2 = (1+y)^{1/2} - 1$, or $y^S = ((1+0.06)^{(1/2)} - 1) \times 2 = 0.0591 = 5.91\%$. It can be also transformed into a continuously compounded rate, using $\exp(y^C) = (1+y)$, or $y^C = \ln(1+0.06) = 0.0583 = 5.83\%$.

Note that as we increase the frequency of the compounding, the resulting rate decreases. Intuitively, because our money works harder with more frequent compounding, a lower investment rate will achieve the same payoff at the end.

KEY CONCEPT

For fixed present value and cash flows, increasing the frequency of the compounding will decrease the associated yield.

EXAMPLE 1.1: FRM EXAM 2002—QUESTION 48

An investor buys a Treasury bill maturing in 1 month for \$987. On the maturity date the investor collects \$1000. Calculate effective annual rate (EAR)

a. 17.0%
b. 15.8%
c. 13.0%
d. 11.6%

EXAMPLE 1.2: FRM EXAM 2002—QUESTION 51

Consider a savings account that pays an annual interest rate of 8%. Calculate the amount of time it would take to double your money. Round to the nearest year.

a. 7 years
b. 8 years
c. 9 years
d. 10 years

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EXAMPLE 1.3: FRM EXAM 1999—QUESTION 17

Assume a semiannual compounded rate of 8% per annum. What is the equivalent annually compounded rate?

a. 9.20%
b. 8.16%
c. 7.45%
d. 8.00%

1.2 PRICE-YIELD RELATIONSHIP

1.2.1 Valuation

The fundamental discounting relationship from Equation (1.1) can be extended to any bond with a fixed cash-flow pattern. We can write the present value of a bond P as the discounted value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{1.6}$$

where:

C_t = the cash flow (coupon or principal) in period t
t = the number of periods (e.g., half-years) to each payment
T = the number of periods to final maturity
y = the discounting factor per period (e.g., $y^S/2$)

A typical cash-flow pattern consists of a fixed coupon payment plus the repayment of the principal, or **face value** at expiration. Define c as the coupon *rate* and F as the face value. We have $C_t = cF$ prior to expiration, and at expiration, we have $C_T = cF + F$. The appendix reviews useful formulas that provide closed-form solutions for such bonds.

When the coupon rate c precisely matches the yield y, using the same compounding frequency, the present value of the bond must be equal to the face value. The bond is said to be a **par bond**.

Equation (1.6) describes the relationship between the yield y and the value of the bond P, given its cash-flow characteristics. In other words, the value P can also be written as a nonlinear function of the yield y:

$$P = f(y) \tag{1.7}$$

FIGURE 1.1 Price-Yield Relationship

Conversely, we can set P to the current market price of the bond, including any accrued interest. From this, we can compute the "implied" yield that will solve this equation.

Figure 1.1 describes the price-yield function for a 10-year bond with a 6% annual coupon. In risk management terms, this is also the relationship between the payoff on the asset and the risk factor. At a yield of 6%, the price is at par, $P =$ \$100. Higher yields imply lower prices. This is an example of a **payoff function**, which links the price to the underlying risk factor.

Over a wide range of yield values, this is a highly nonlinear relationship. For instance, when the yield is zero, the value of the bond is simply the sum of cash flows, or \$160 in this case. When the yield tends to very large values, the bond price tends to zero. For small movements around the initial yield of 6%, however, the relationship is quasilinear.

There is a particularly simple relationship for **consols**, or **perpetual bonds**, which are bonds making regular coupon payments but with no redemption date. For a consol, the maturity is infinite and the cash flows are all equal to a fixed percentage of the face value, $C_t = C = cF$. As a result, the price can be simplified from Equation (1.6) to

$$P = cF\left[\frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] = \frac{c}{y}F \tag{1.8}$$

as shown in the appendix. In this case, the price is simply proportional to the inverse of the yield. Higher yields lead to lower bond prices, and vice versa.

Example: Valuing a Bond

Consider a bond that pays \$100 in 10 years and a 6% annual coupon. Assume that the next coupon payment is in exactly one year. What is the market value if the yield is 6%? If it falls to 5%?

The bond cash flows are $C_1 = \$6$, $C_2 = \$6, \ldots, C_{10} = \$106$. Using Equation (1.6) and discounting at 6%, this gives the present value of cash flows of \$5.66, \$5.34, ..., \$59.19, for a total of \$100.00. The bond is selling at par. This is logical because the coupon is equal to the yield, which is also annually compounded. Alternatively, discounting at 5% leads to a price of \$107.72.

EXAMPLE 1.4: FRM EXAM 1998—QUESTION 12

A fixed-rate bond, currently priced at 102.9, has one year remaining to maturity and is paying an 8% coupon. Assuming the coupon is paid semiannually, what is the yield of the bond?

a. 8%
b. 7%
c. 6%
d. 5%

1.2.2 Taylor Expansion

Let us say that we want to see what happens to the price if the yield changes from its initial value, called y_0, to a new value, $y_1 = y_0 + \Delta y$. Risk management is all about assessing the effect of changes in risk factors such as yields on asset values. Are there shortcuts to help us with this?

We could recompute the new value of the bond as $P_1 = f(y_1)$. If the change is not too large, however, we can apply a very useful shortcut. The nonlinear relationship can be approximated by a **Taylor expansion** around its initial value[1]

$$P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2} f''(y_0)(\Delta y)^2 + \cdots \tag{1.9}$$

where $f'(\cdot) = \frac{dP}{dy}$ is the first derivative and $f''(\cdot) = \frac{d^2P}{dy^2}$ is the second derivative of the function $f(\cdot)$ valued at the starting point.[2] This expansion can be generalized

[1] This is named after the English mathematician Brook Taylor (1685–1731), who published this result in 1715. The full recognition of the importance of this result only came in 1755 when Euler applied it to differential calculus.

[2] This first assumes that the function can be written in polynomial form as $P(y + \Delta y) = a_0 + a_1\Delta y + a_2(\Delta y)^2 + \cdots$, with unknown coefficients a_0, a_1, a_2. To solve for the first, we set $\Delta y = 0$. This gives $a_0 = P_0$. Next, we take the derivative of both sides and set $\Delta y = 0$. This gives $a_1 = f'(y_0)$. The next step gives $2a_2 = f''(y_0)$. Here, the term "derivatives" takes the usual mathematical interpretation, and has nothing to do with *derivatives products* such as options.

to situations where the function depends on two or more variables. For bonds, the first derivative is related to the *duration* measure, and the second to *convexity*.

Equation (1.9) represents an infinite expansion with increasing powers of Δy. Only the first two terms (linear and quadratic) are ever used by finance practitioners. They provide a good approximation to changes in prices relative to other assumptions we have to make about pricing assets. If the increment is very small, even the quadratic term will be negligible.

Equation (1.9) is fundamental for risk management. It is used, sometimes in different guises, across a variety of financial markets. We will see later that this Taylor expansion is also used to approximate the movement in the value of a derivatives contract, such as an option on a stock. In this case, Equation (1.9) is

$$\Delta P = f'(S)\Delta S + \frac{1}{2} f''(S)(\Delta S)^2 + \cdots \tag{1.10}$$

where S is now the price of the underlying asset, such as the stock. Here, the first derivative $f'(S)$ is called *delta*, and the second $f''(S)$, *gamma*.

The Taylor expansion allows easy aggregation across financial instruments. If we have x_i units (numbers) of bond i and a total of N different bonds in the portfolio, the portfolio derivatives are given by

$$f'(y) = \sum_{i=1}^{N} x_i f_i'(y) \tag{1.11}$$

We will illustrate this point later for a three-bond portfolio.

EXAMPLE 1.5: FRM EXAM 1999—QUESTION 9

A number of terms in finance are related to the (calculus!) derivative of the price of a security with respect to some other variable. Which pair of terms is defined using second derivatives?

a. Modified duration and volatility
b. Vega and delta
c. Convexity and gamma
d. PV01 and yield to maturity

1.3 BOND PRICE DERIVATIVES

For fixed-income instruments, the derivatives are so important that they have been given a special name.[3] The negative of the first derivative is the **dollar**

[3] Note that this chapter does not present duration in the traditional textbook order. In line with the advanced focus on risk management, we first analyze the properties of duration as a sensitivity

duration (**DD**):

$$f'(y_0) = \frac{dP}{dy} = -D^* \times P_0 \tag{1.12}$$

where D^* is called the **modified duration**. Thus, dollar duration is

$$DD = D^* \times P_0 \tag{1.13}$$

where the price P_0 represent the *market* price, including any accrued interest. Sometimes, risk is measured as the **dollar value of a basis point** (**DVBP**):

$$DVBP = DD \times \Delta y = [D^* \times P_0] \times 0.0001 \tag{1.14}$$

with 0.0001 representing an interest rate change of one basis point (bp), or one hundredth of a percent. The **DVBP**, sometimes called the **DV01**, measures can be easily added up across the portfolio.

The second derivative is the **dollar convexity** (**DC**):

$$f''(y_0) = \frac{d^2P}{dy^2} = C \times P_0 \tag{1.15}$$

where C is called the **convexity**.

For fixed-income instruments with known cash flows, the price-yield function is known, and we can compute analytical first and second derivatives. Consider, for example, our simple zero-coupon bond in Equation (1.1), where the only payment is the face value, $C_T = F$. We take the first derivative, which is

$$\frac{dP}{dy} = \frac{d}{dy}\left[\frac{F}{(1+y)^T}\right] = (-T)\frac{F}{(1+y)^{T+1}} = -\frac{T}{(1+y)}P \tag{1.16}$$

Comparing with Equation (1.12), we see that the modified duration must be given by $D^* = T/(1+y)$. The conventional measure of **duration** is $D = T$, which does not include division by $(1+y)$ in the denominator. This is also called **Macaulay duration**. Note that duration is expressed in periods, like T. With annual compounding, duration is in years. With semiannual compounding, duration is in semesters. It then has to be divided by two for conversion to years. Modified duration D^* is related to Macaulay duration D

$$D^* = \frac{D}{(1+y)} \tag{1.17}$$

Modified duration is the appropriate measure of interest rate exposure. The quantity $(1+y)$ appears in the denominator because we took the derivative of the present value term with discrete compounding. If we use continuous compounding, modified duration is identical to the conventional duration measure. In practice, the difference between Macaulay and modified duration is usually small.

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measure. This applies to any type of fixed-income instrument. Later, we will illustrate the usual definition of duration as a weighted average maturity, which applies for fixed-coupon bonds only.

Let us now go back to Equation (1.16) and consider the second derivative, which is

$$\frac{d^2P}{dy^2} = -(T+1)(-T)\frac{F}{(1+y)^{T+2}} = \frac{(T+1)T}{(1+y)^2} \times P \tag{1.18}$$

Comparing with Equation (1.15), we see that the convexity is $C = (T+1)T/(1+y)^2$. Note that its dimension is expressed in period squared. With semiannual compounding, convexity is measured in semesters squared. It then has to be divided by 4 for conversion to years squared.[4] So, convexity must be positive for bonds with fixed coupons.

Putting together all these equations, we get the Taylor expansion for the change in the price of a bond, which is

$$\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2 + \cdots \tag{1.19}$$

Therefore duration measures the first-order (linear) effect of changes in yield and convexity the second-order (quadratic) term.

Example: Computing the Price Approximation*

Consider a 10-year zero-coupon Treasury bond trading at a yield of 6%. The present value is obtained as $P = 100/(1+6/200)^{20} = 55.368$. As is the practice in the Treasury market, yields are semiannually compounded. Thus, all computations should be carried out using semesters, after which final results can be converted into annual units.

Here, Macaulay duration is exactly 10 years, as $D = T$ for a zero coupon bond. Its modified duration is $D^* = 20/(1+6/200) = 19.42$ semesters, which is 9.71 years. Its convexity is $C = 21 \times 20/(1+6/200)^2 = 395.89$ semesters squared, which is 98.97 in years squared. $DD = D^* \times P = 9.71 \times \$55.37 = \$537.55$. $DVBP = DD \times 0.0001 = \$0.0538$.

We want to approximate the change in the value of the bond if the yield goes to 7%. Using Equation (1.19), we have $\Delta P = -[9.71 \times \$55.37](0.01) + 0.5[98.97 \times \$55.37](0.01)^2 = -\$5.375 + \$0.274 = -\$5.101$. Using the linear term only, the new price is $\$55.368 - \$5.375 = \$49.992$. Using the two terms in the expansion, the predicted price is slightly higher, at $\$55.368 - \$5.375 + \$0.274 = \$50.266$.

These numbers can be compared with the exact value, which is \$50.257. The linear approximation has a relative pricing error of −0.53%, which is not bad. Adding a quadratic term reduces this to an error of 0.02% only, which is very small, given typical bid-ask spreads.

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[4] This is because the conversion to annual terms is obtained by multiplying the semiannual yield Δy by two. As a result, the duration term must be divided by 2 and the convexity term by 2^2, or 4, for conversion to annual units.

*For such examples in this handbook, please note that intermediate numbers are reported with fewer significant digits than actually used in the computations. As a result, using rounded off numbers may give results that differ sligthly from the final numbers shown here.

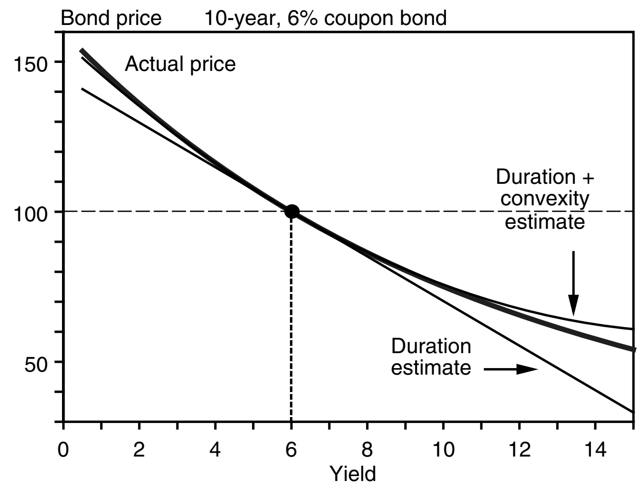

FIGURE 1.2 Price Approximation

More generally, Figure 1.2 compares the quality of the Taylor series approximation. We consider a 10-year bond paying a 6% coupon semiannually. Initially, the yield is also at 6% and, as a result, the price of the bond is at par, at \$100. The graph compares three lines representing the following:

1. The actual, exact price $P = f(y_0 + \Delta y)$
2. The duration estimate $P = P_0 - D^* P_0 \Delta y$
3. The duration and convexity estimate $P = P_0 - D^* P_0 \Delta y + (1/2) C P_0 (\Delta y)^2$

The actual price curve shows an increase in the bond price if the yield falls and, conversely, a depreciation if the yield increases. This effect is captured by the tangent to the true price curve, which represents the linear approximation based on duration. For small movements in the yield, this linear approximation provides a reasonable fit to the exact price.

KEY CONCEPT

Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point.

For large movements in price, however, the price-yield function becomes more curved and the linear fit deteriorates. Under these conditions, the quadratic approximation is noticeably better.

We should also note that the curvature is away from the origin, which explains the term *convexity* (as opposed to *concavity*). Figure 1.3 compares curves with different values for convexity. This curvature is beneficial because the second-order effect $0.5[C \times P](\Delta y)^2$ *must* be positive when convexity is positive.

As Figure 1.3 shows, when the yield rises, the price drops but less than predicted by the tangent. Conversely, if the yield falls, the price increases faster than along the tangent. In other words, the quadratic term is always beneficial.

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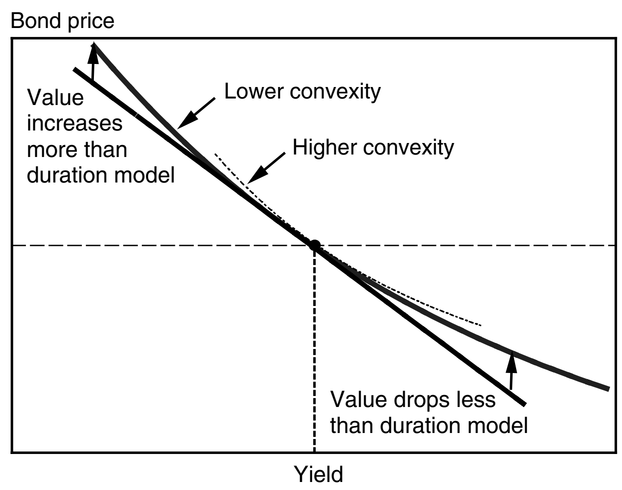

FIGURE 1.3 Effect of Convexity

KEY CONCEPT

Convexity is always positive for regular coupon-paying bonds. Greater convexity is beneficial both for falling and rising yields.

The bond's modified duration and convexity can also be computed directly from numerical derivatives. Duration and convexity cannot be computed directly for some bonds, such as mortgage-backed securities, because their cash flows are uncertain. Instead, the portfolio manager has access to pricing models that can be used to reprice the securities under various yield environments.

We choose a change in the yield, Δy, and reprice the bond under an upmove scenario, $P_+ = P(y_0 + \Delta y)$, and downmove scenario, $P_- = P(y_0 - \Delta y)$. **Effective duration** is measured by the numerical derivative. Using $D^* = -(1/P)dP/dy$, it is estimated as

$$D^E = \frac{[P_- - P_+]}{(2P_0 \Delta y)} = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2\Delta y)P_0} \tag{1.20}$$

Using $C = (1/P)d^2P/dy^2$, **effective convexity** is estimated as

$$C^E = [D_- - D_+]/\Delta y = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0 \Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0 \Delta y)}\right] / \Delta y \tag{1.21}$$

To illustrate, consider a 30-year zero-coupon bond with a yield of 6%, semiannually compounded. The initial price is \$16.9733. We revalue the bond at 5% and 7%, with prices shown in Table 1.1. The effective duration in Equation (1.20) uses the two extreme points. The effective convexity in Equation (1.21) uses the difference between the dollar durations for the upmove and downmove. Note that convexity is positive if duration increases as yields fall, or if $D_- > D_+$.

TABLE 1.1 Effective Duration and Convexity

State	Yield (%)	Bond Value	Duration Computation	Convexity Computation
Initial y_0	6.00	16.9733		
Up $y_0 + \Delta y$	7.00	12.6934		Duration up: 25.22
Down $y_0 - \Delta y$	5.00	22.7284		Duration down: 33.91
Difference in values			−10.0349	8.69
Difference in yields			0.02	0.01
Effective measure			29.56	869.11
Exact measure			29.13	862.48

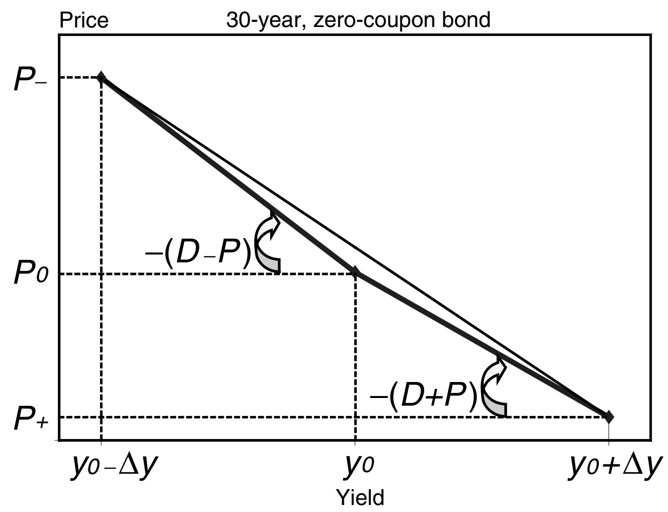

FIGURE 1.4 Effective Duration and Convexity

The computations are detailed in Table 1.1, which shows an effective duration of 29.56. This is very close to the true value of 29.13, and would be even closer if the step Δy was smaller. Similarly, the effective convexity is 869.11, which is close to the true value of 862.48.

Finally, this numerical approach can be applied to get an estimate of the duration of a bond by considering bonds with the same maturity but different coupons. If interest rates decrease by 1%, the market price of a 6% bond should go up to a value close to that of a 7% bond. Thus, we replace a drop in yield of Δy with an increase in coupon Δc and use the effective duration method to find the **coupon curve duration:**[5]

$$D^{CC} = \frac{[P_+ - P_-]}{(2P_0\Delta c)} = \frac{P(y_0; c + \Delta c) - P(y_0; c - \Delta c)}{(2\Delta c)P_0} \tag{1.22}$$

This approach is useful for securities that are difficult to price under various yield scenarios. It requires only the market prices of securities with different coupons.

[5] For an example of a more formal proof, we could take the pricing formula for a consol at par and compute the derivatives with respect to y and c. Apart from the sign, these derivatives are identical when $y = c$.

Example: Computation of Coupon Curve Duration

Consider a 10-year bond that pays a 7% coupon semiannually. In a 7% yield environment, the bond is selling at par and has modified duration of 7.11 years. The prices of 6% and 8% coupon bonds are \$92.89 and \$107.11, respectively. This gives a coupon curve duration of $(107.11 - 92.89)/(0.02 \times 100) = 7.11$, which in this case is the same as modified duration.

EXAMPLE 1.6: FRM EXAM 2004—QUESTION 44

Consider a 2-year, 6% semi-annual bond currently yielding 5.2% on a bond equivalent basis. If the Macaulay duration of the bond is 1.92 years, its modified duration is closest to

a. 1.97 years
b. 1.78 years
c. 1.87 years
d. 2.04 years

EXAMPLE 1.7: FRM EXAM 1998—QUESTION 22

What is the price impact of a 10-basis-point increase in yield on a 10-year par bond with a modified duration of 7 and convexity of 50?

a. −0.705
b. −0.700
c. −0.698
d. −0.690

EXAMPLE 1.8: FRM EXAM 1998—QUESTION 17

A bond is trading at a price of 100 with a yield of 8%. If the yield increases by 1 basis point, the price of the bond will decrease to 99.95. If the yield decreases by 1 basis point, the price of the bond will increase to 100.04. What is the modified duration of the bond?

a. 5.0
b. 5.0
c. 4.5
d. −4.5

EXAMPLE 1.9: FRM EXAM 1998—QUESTION 20

Coupon curve duration is a useful method to estimate duration from market prices of a mortgage-backed security (MBS). Assume the coupon curve of prices for Ginnie Maes in June 2001 is as follows: 6% at 92, 7% at 94, and 8% at 96.5. What is the estimated duration of the 7s?

a. 2.45
b. 2.40
c. 2.33
d. 2.25

1.3.1 Interpreting Duration and Convexity

The preceding section has shown how to compute analytical formulas for duration and convexity in the case of a simple zero-coupon bond. We can use the same approach for coupon-paying bonds. Going back to Equation (1.6), we have

$$\frac{dP}{dy} = \sum_{t=1}^{T} \frac{-tC_t}{(1+y)^{t+1}} = -\left[\sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}\right] / P \times \frac{P}{(1+y)} = -\frac{D}{(1+y)} P \tag{1.23}$$

which defines duration as

$$D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t} / P \tag{1.24}$$

The economic interpretation of duration is that it represents the average time to wait for each payment, weighted by the present flow. Indeed, replacing P, we can write write

$$D = \sum_{t=1}^{T} t \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} t \times w_t \tag{1.25}$$

where the weights w_t represent the ratio of the present value of each cash flow C_t relative to the total, and sum to unity. This explains why the duration of a zero-coupon bond is equal to the maturity. There is only one cash flow, and its weight is one.

KEY CONCEPT

(Macaulay) duration represents an average of the time to wait for all cash flows.

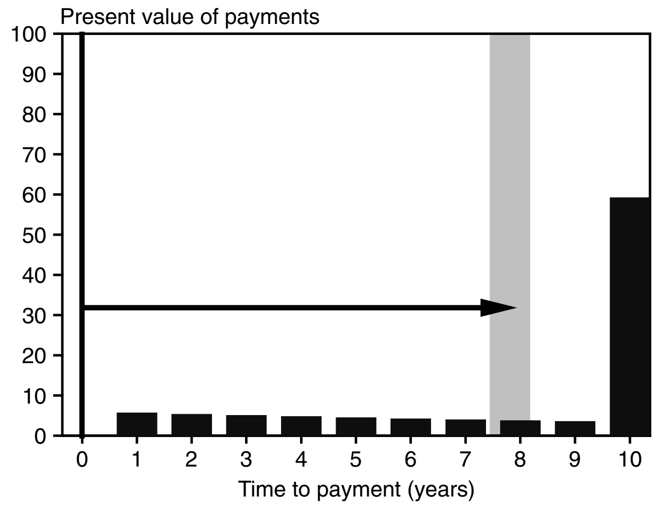

FIGURE 1.5 Duration as the Maturity of a Zero-Coupon Bond

Figure 1.5 lays out the present value of the cash flows of a 6% coupon, 10-year bond. Given a duration of 7.80 years, this coupon-paying bond is equivalent to a zero-coupon bond maturing in exactly 7.80 years.

For bonds with fixed coupons, duration is less than maturity. For instance, Figure 1.6 shows how the duration of a 10-year bond varies with its coupon. With a zero coupon, Macaulay duration is equal to maturity. Higher coupons place more weight on prior payments and therefore reduce duration.

Duration can be expressed in a simple form for **consols**. From Equation (1.8), we have $P = (c/y)F$. Taking the derivative, we find

$$\frac{dP}{dy} = cF\frac{(-1)}{y^2} = (-1)\frac{1}{y}\left[\frac{c}{y}F\right] = (-1)\frac{1}{y}P = -\frac{D_C}{(1+y)}P \tag{1.26}$$

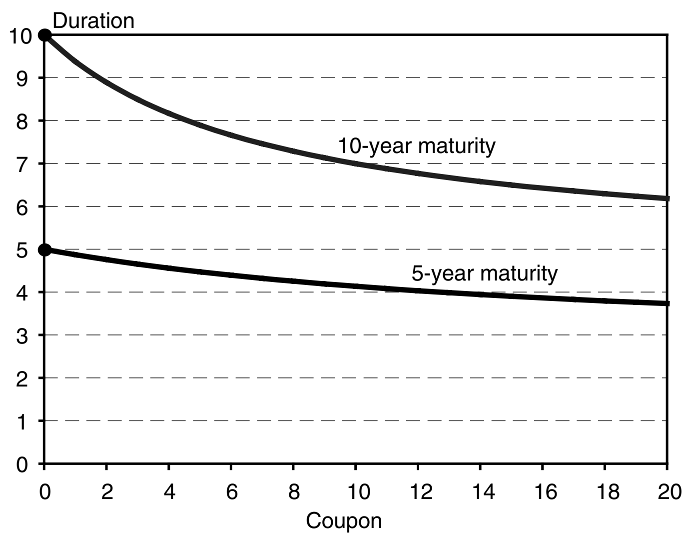

FIGURE 1.6 Duration and Coupon

Hence, the Macaulay duration for the consol D_C is

$$D_C = \frac{(1+y)}{y} \tag{1.27}$$

This shows that the duration of a consol is finite even if its maturity is infinite. Also, this duration does not depend on the coupon.

This formula provides a useful rule of thumb. For a long-term coupon-paying bond, duration should be lower than $(1+y)/y$. For instance, when $y = 6\%$, the upper limit on duration is $D_C = 1.06/0.06$, or 17.7 years. In this environment, the duration of a par 30-year bond is 14.25, which is indeed lower than 17.7 years.

KEY CONCEPT

The duration of a long-term bond can be approximated by an upper bound, which is that of a consol with the same yield, $D_C = (1+y)/y$.

Figure 1.7 describes the relationship between duration, maturity, and coupon for regular bonds in a 6% yield environment. For the zero-coupon bond, $D = T$, which is a straight line going through the origin. For the par 6% bond, duration increases monotonically with maturity until it reaches the asymptote of D_C. The 8% bond has lower duration than the 6% bond for fixed T. Greater coupons, for a fixed maturity, decrease duration, as more of the payments come early.

Finally, the 2% bond displays a pattern intermediate between the zero-coupon and 6% bonds. It initially behaves like the zero, exceeding D_C initially and then falling back to the asymptote, which is the same for all coupon-paying bonds.

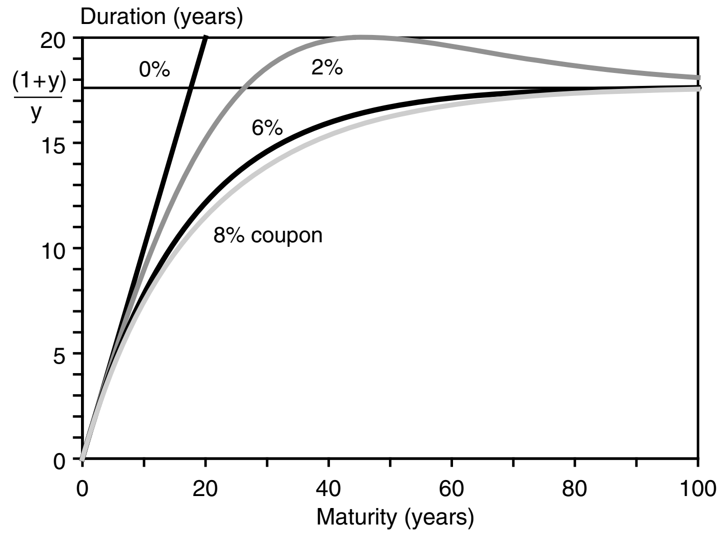

FIGURE 1.7 Duration and Maturity

Taking now the second derivative in Equation (1.23), we have

$$\frac{d^2P}{dy^2} = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} = \left[\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}}/P\right] \times P \tag{1.28}$$

which defines convexity as

$$C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}}/P \tag{1.29}$$

Convexity can also be written as

$$C = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times w_t \tag{1.30}$$

Because the squared t term dominates in the fraction, this basically involves a weighted average of the square of time. Therefore, convexity is much greater for long-maturity bonds because they have payoffs associated with large values of t. The formula also shows that convexity is always positive for such bonds, implying that the curvature effect is beneficial. As we will see later, convexity can be negative for bonds that have uncertain cash flows, such as **mortgage-backed securities** (MBSs) or callable bonds.

Figure 1.8 displays the behavior of convexity, comparing a zero-coupon bond with a 6% coupon bond with identical maturities. The zero-coupon bond always has greater convexity, because there is only one cash flow at maturity. Its convexity is roughly the square of maturity, for example about 900 for the 30-year zero. In contrast, the 30-year coupon bond has a convexity of about 300 only.

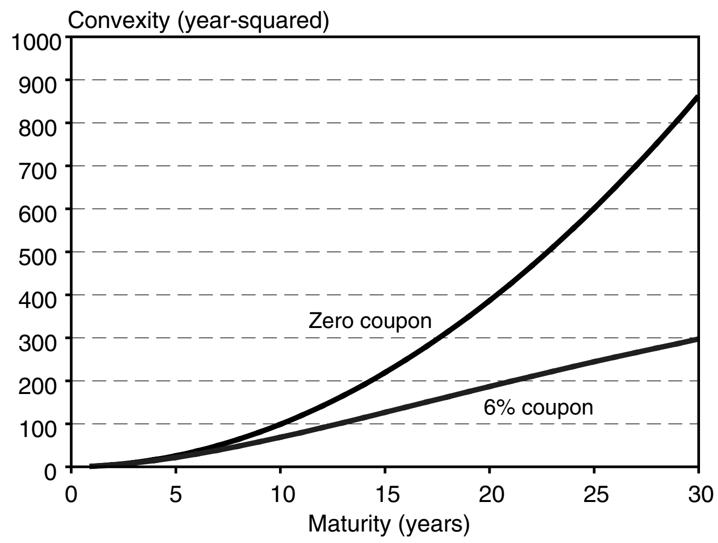

FIGURE 1.8 Convexity and Maturity

As an illustration, Table 1.2 details the steps of the computation of duration and convexity for a two-year, 6% semiannual coupon-paying bond. We first convert the annual coupon and yield into semiannual equivalent, \$3 and 3% each. The PV column then reports the present value of each cash flow. We verify that these add up to \$100, since the bond must be selling at par.

Next, the duration term column multiplies each PV term by time, or, more precisely, the number of half years until payment. This adds up to \$382.86, which, divided by the price gives $D = 3.83$. This number is measured in half years, and we need to divide by two to convert to years. Macaulay duration is 1.91 years, and modified duration $D^* = 1.91/1.03 = 1.86$ years. Note that, to be consistent, the adjustment in the denominator involves the semiannual yield of 3%.

Finally, the right-most column shows how to compute the bond's convexity. Each term involves PV_t times $t(t+1)/(1+y)^2$. These terms sum to 1,777.755, or divided by the price, 17.78. This number is expressed in units of time squared and must be divided by 4 to be converted in annual terms. We find a convexity of $C = 4.44$, in year-squared.

TABLE 1.2 Computing Duration and Convexity

Period (half-year) t	Payment C_t	Yield (%) (6 mo)	PV of Payment $C_t/(1+y)^t$	Duration Term tPV_t	Convexity Term $t(t+1)PV_t$ $\times(1/(1+y)^2)$
1	3	3.00	2.913	2.913	5.491
2	3	3.00	2.828	5.656	15.993
3	3	3.00	2.745	8.236	31.054
4	103	3.00	91.514	366.057	1725.218
Sum:			100.00	382.861	1777.755
(half-years)				3.83	17.78
(years)				1.91	
Modified duration				1.86	
Convexity					4.44

EXAMPLE 1.10: FRM EXAM 2003—QUESTION 13

Suppose the face value of a three-year option-free bond is USD 1,000 and the annual coupon is 10%. The current yield to maturity is 5%. What is the modified duration of this bond?

- **a.** 2.62
- **b.** 2.85
- **c.** 3.00
- **d.** 2.75

EXAMPLE 1.11: FRM EXAM 2002—QUESTION 118

A Treasury bond has a coupon rate of 6% per annum (the coupons are paid semiannually) and a semiannually compounded yield of 4% per annum. The bond matures in 18 months and the next coupon will be paid 6 months from now. Which number below is closest to the bond's Macaulay duration?

a. 1.023 years
b. 1.457 years
c. 1.500 years
d. 2.915 years

EXAMPLE 1.12: FRM EXAM 1998—QUESTION 29

A and B are two perpetual bonds, that is, their maturities are infinite. A has a coupon of 4% and B has a coupon of 8%. Assuming that both are trading at the same yield, what can be said about the duration of these bonds?

a. The duration of A is greater than the duration of B.
b. The duration of A is less than the duration of B.
c. A and B both have the same duration.
d. None of the above.

EXAMPLE 1.13: FRM EXAM 1997—QUESTION 24

Which of the following is *not* a property of bond duration?

a. For zero-coupon bonds, Macaulay duration of the bond equals its years to maturity.
b. Duration is usually inversely related to the coupon of a bond.
c. Duration is usually higher for higher yields to maturity.
d. Duration is higher as the number of years to maturity for a bond selling at par or above increases.

EXAMPLE 1.14: FRM EXAM 2004—QUESTION 16

A manager wants to swap a bond for a bond with the same price but higher duration. Which of the following bond characteristics would be associated with a higher duration?

I. A higher coupon rate
II. More frequent coupon payments
III. A longer term to maturity
IV. A lower yield

a. I, II, and III
b. II, III, and IV
c. III and IV
d. I and II

EXAMPLE 1.15: FRM EXAM 2001—QUESTION 104

When the maturity of a plain coupon bond increases, its duration increases

a. Indefinitely and regularly
b. Up to a certain level
c. Indefinitely and progressively
d. In a way dependent on the bond being priced above or below par

EXAMPLE 1.16: FRM EXAM 2000—QUESTION 106

Consider the following bonds:

Bond Number	Maturity (yrs)	Coupon Rate	Frequency	Yield (Annual)
1	10	6%	1	6%
2	10	6%	2	6%
3	10	0%	1	6%
4	10	6%	1	5%
5	9	6%	1	6%

How would you rank the bonds from the shortest to longest duration?

a. 5-2-1-4-3
b. 1-2-3-4-5
c. 5-4-3-1-2
d. 2-4-5-1-3

1.3.2 Portfolio Duration and Convexity

Fixed-income portfolios often involve very large numbers of securities. It would be impractical to consider the movements of each security individually. Instead, portfolio managers aggregate the duration and convexity across the portfolio. A manager who believes that rates will increase should shorten the portfolio duration relative to that of the benchmark. Say, for instance, that the benchmark has a duration of five years. The manager shortens the portfolio duration to one year only. If rates increase by 2%, the benchmark will lose approximately $5y \times 2\% = 10\%$. The portfolio, however, will only lose $1y \times 2\% = 2\%$, hence "beating" the benchmark by 8%.

Because the Taylor expansion involves a summation, the portfolio duration is easily obtained from the individual components. Say we have N components indexed by i. Defining D_p^* and P_p as the portfolio modified duration and value, the portfolio dollar duration (DD) is

$$D_p^* P_p = \sum_{i=1}^{N} D_i^* x_i P_i \tag{1.31}$$

where x_i is the number of units of bond i in the portfolio. A similar relationship holds for the portfolio dollar convexity (DC). If yields are the same for all components, this equation also holds for the Macaulay duration.

Because the portfolio's total market value is simply the summation of the component market values,

$$P_p = \sum_{i=1}^{N} x_i P_i \tag{1.32}$$

we can define the **portfolio weight** w_i as $w_i = x_i P_i / P_p$, provided that the portfolio market value is nonzero. We can then write the portfolio duration as a weighted average of individual durations

$$D_p^* = \sum_{i=1}^{N} D_i^* w_i \tag{1.33}$$

Similarly, the portfolio convexity is a weighted average of convexity numbers

$$C_p = \sum_{i=1}^{N} C_i w_i \tag{1.34}$$

As an example, consider a portfolio invested in three bonds, described in Table 1.3. The portfolio is long a 10-year and 1-year bond, and short a 30-year zero-coupon bond. Its market value is \$1,301,600. Summing the duration for each component, the portfolio dollar duration is \$2,953,800, which translates into a duration of 2.27 years. The portfolio convexity is $-76{,}918{,}323/1{,}301{,}600 = -59.10$,

TABLE 1.3 Portfolio Dollar Duration and Convexity

	Bond 1	Bond 2	Bond 3	Portfolio
Maturity (years)	10	1	30	
Coupon	6%	0%	0%	
Yield	6%	6%	6%	
Price P_i	\$100.00	\$94.26	\$16.97	
Modified duration D_i^*	7.44	0.97	29.13	
Convexity C_i	68.78	1.41	862.48	
Number of bonds x_i	10,000	5,000	−10,000	
Dollar amounts $x_i P_i$	\$1,000,000	\$471,300	−\$169,700	\$1,301,600
Weight w_i	76.83%	36.21%	−13.04%	100.00%
Dollar duration $D_i^* P_i$	\$744.00	\$91.43	\$494.34	
Portfolio DD: $x_i D_i^* P_i$	\$7,440,000	\$457,161	−\$4,943,361	\$2,953,800
Portfolio DC: $x_i C_i P_i$	68,780,000	664,533	−146,362,856	−76,918,323

which is negative due to the short position in the 30-year zero, which has very high convexity.

Alternatively, assume the portfolio manager is given a benchmark that is the first bond. He or she wants to invest in bonds 2 and 3, keeping the portfolio duration equal to that of the target, or 7.44 years. To achieve the target value and dollar duration, the manager needs to solve a system of two equations in the numbers x_1 and x_2:

$$\text{Value: } \$100 = x_1\$94.26 + x_2\$16.97$$

$$\text{Dol. Duration: } 7.44 \times \$100 = 0.97 \times x_1\$94.26 + 29.13 \times x_2\$16.97$$

The solution is $x_1 = 0.817$ and $x_2 = 1.354$, which gives a portfolio value of \$100 and modified duration of 7.44 years.[6] The portfolio convexity is 199.25, higher than the index. Such a portfolio consisting of very short and very long maturities is called a **barbell portfolio**. In contrast, a portfolio with maturities in the same range is called a **bullet portfolio**. Note that the barbell portfolio has a much greater convexity than the bullet bond because of the payment in 30 years. Such a portfolio would be expected to outperform the bullet portfolio if yields moved by a large amount.

In sum, duration and convexity are key measures of fixed-income portfolios. They summarize the linear and quadratic exposure to movements in yields. This explains why they are essential tools for fixed-income portfolio managers.

[6] This can be obtained by first expressing x_2 in the first equation as a function of x_1 and then substituting back into the second equation. This gives $x_2 = (100 - 94.26x_1)/16.97$, and $744 = 91.43x_1 + 494.34x_2 = 91.43x_1 + 494.34(100 - 94.26x_1)/16.97 = 91.43x_1 + 2913.00 - 2745.79x_1$. Solving, we find $x_1 = (-2169.00)/(-2654.36) = 0.817$ and $x_2 = (100 - 94.26 \times 0.817)/16.97 = 1.354$.

EXAMPLE 1.17: FRM EXAM 2002—QUESTION 57

A bond portfolio has the following composition:

1. Portfolio A: price \$90,000, modified duration 2.5, long position in 8 bonds
2. Portfolio B: price \$110,000, modified duration 3, short position in 6 bonds
3. Portfolio C: price \$120,000, modified duration 3.3, long position in 12 bonds

All interest rates are 10%. If the rates rise by 25 basis points, then the bond portfolio value will

a. Decrease by \$11,430
b. Decrease by \$21,330
c. Decrease by \$12,573
d. Decrease by \$23,463

EXAMPLE 1.18: FRM EXAM 2000—QUESTION 110

Which of the following statements are *true*?

I. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 10-year, 6% bond.
II. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 6% bond with a duration of 10 years.
III. Convexity grows proportionately with the maturity of the bond.
IV. Convexity is always positive for all types of bonds.
V. Convexity is always positive for "straight" bonds.

a. I only
b. I and II only
c. I and V only
d. II, III, and V only

1.4 IMPORTANT FORMULAS

Compounding: $(1 + y)^T = (1 + y^S/2)^{2T} = e^{y^C T}$
Fixed-coupon bond valuation: $P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$
Taylor expansion: $P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2} f''(y_0)(\Delta y)^2 + \cdots$
Duration as exposure: $\frac{dP}{dy} = -D^* \times P$, $\mathrm{DD} = D^* \times P$, $\mathrm{DVBP} = \mathrm{DD} \times 0.0001$

Conventional duration: $D^* = \frac{D}{(1+y)}$, $D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}/P$
Convexity: $\frac{d^2P}{dy^2} = C \times P$, $C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}}/P$
Price change: $\Delta P = -[D^* \times P](\Delta y) + 0.5[C \times P](\Delta y)^2 + \cdots$
Consol: $P = \frac{c}{y}F$, $D = \frac{(1+y)}{y}$
Portfolio duration and convexity: $D_p^* = \sum_{i=1}^{N} D_i^* w_i$, $C_p = \sum_{i=1}^{N} C_i w_i$

1.5 ANSWERS TO CHAPTER EXAMPLES

Example 1.1: FRM Exam 2002—Question 48

a. The EAR is defined by $FV/PV = (1 + \text{EAR})^T$. So $\text{EAR} = (FV/PV)^{1/T} - 1$. Here, $T = 1/12$. So, $\text{EAR} = (1{,}000/987)^{12} - 1 = 17.0\%$.

Example 1.2: FRM Exam 2002—Question 51

c. The time T relates the current and future values such that $FV/PV = 2 = (1 + 8\%)^T$. Taking logs of both sides, this gives $T = \ln(2)/\ln(1.08) = 9.006$.

Example 1.3: FRM Exam 1999—Question 17

b. This is derived from $(1 + y^S/2)^2 = (1 + y)$, or $(1 + 0.08/2)^2 = 1.0816$, which gives 8.16%. This makes sense because the annual rate must be higher due to the less frequent compounding.

Example 1.4: FRM Exam 1998—Question 12

d. We need to find y such that $\$4/(1 + y/2) + \$104/(1 + y/2)^2 = \$102.90$. Solving, we find $y = 5\%$. (This can be computed on a HP-12C calculator, for example.) There is another method for finding y. This bond has a duration of about one year, implying that, approximately, $\Delta P = -1 \times \$100 \times \Delta y$. If the yield was 8%, the price would be at \$100. Instead, the change in price is $\Delta P = \$102.90 - \$100 = \$2.90$. Solving for Δy, the change in yield must be approximately -3%, leading to $8 - 3 = 5\%$.

Example 1.5: FRM Exam 1999—Question 9

c. First derivatives involve modified duration and delta. Second derivatives involve convexity (for bonds) and gamma (for options).

Example 1.6: FRM Exam 2004—Question 44

c. Modified duration is given by $D/(1 + y)$, using the appropriate compounding frequency for the denominator, which is semi-annual. Therefore, $D^* = 1.92/(1 + 0.052/2) = 1.87$. This makes sense because modified duration is slightly below Macaulay duration.

Example 1.7: FRM Exam 1998—Question 22

c. Since this is a par bond, the initial price is $P = \$100$. The price impact is $\Delta P = -D^* P \Delta y + (1/2)CP(\Delta y)^2 = -(7 \times \$100)(0.001) + (1/2)(50 \times \$100)(0.001)^2 = -0.70 + 0.0025 = -0.6975$. The price falls slightly less than predicted by duration alone.

Example 1.8: FRM Exam 1998—Question 17

c. This question deals with effective duration, which is obtained from full repricing of the bond with an increase and a decrease in yield. This gives a modified duration of $D^* = -(\Delta P/\Delta y)/P = -((99.95 - 100.04)/0.0002)/100 = 4.5$.

Example 1.9: FRM Exam 1998—Question 20

b. The initial price of the 7s is 94. The price of the 6s is 92; this lower coupon is roughly equivalent to an upmove of $\Delta y = 0.01$. Similarly, the price of the 8s is 96.5; this higher coupon is roughly equivalent to a downmove of $\Delta y = 0.01$. The effective modified duration is then $D^E = (P_- - P_+)/(2\Delta y P_0) = (96.5 - 92)/(2 \times 0.01 \times 94) = 2.394$.

Note that we can also compute effective convexity. Modified duration in the downstate is $D_- = (P_- - P_0)/(\Delta y P_0) = (96.5 - 94)/(0.01 \times 94) = 2.6596$. Similarly, the modified duration for an upmove is $D_+ = (P_0 - P_+)/(\Delta y P_0) = (94 - 92)/(0.01 \times 94) = 2.1277$. Convexity is $C^E = (D_- - D_+)/(\Delta y) = (2.6596 - 2.1277)/0.01 = 53.19$.

Example 1.10: FRM Exam 2003—Question 13

d. As in Table 1.2, we lay out the cash flows and find

Period t	Payment C_t	Yield y	$PV_t = C_t/(1+y)^t$	tPV_t
1	100	5.00	95.24	95.24
2	100	5.00	90.71	181.41
3	1100	5.00	950.22	2850.66
Sum:			1136.16	3127.31

Duration is then 2.75, and modified duration 2.62.

Example 1.11: FRM Exam 2002—Question 118

b. For coupon-paying bonds, Macaulay duration is slightly less maturity, which is 1.5 year here. So, b) would be a good guess. guess. Otherwise, we can compute duration exactly.

Example 1.12: FRM Exam 1998—Question 29

c. Going back to the duration equation for the consol, Equation (1.27), we see that it does not depend on the coupon but only on the yield. Hence, the durations must be the same. The price of bond A, however, must be half that of bond B.

Example 1.13: FRM Exam 1997—Question 24

c. Duration usually increases as the time to maturity increases (Figure 1.7), so d) is correct. Macaulay duration is also equal to maturity for zero-coupon bonds, so a) is correct. Figure 1.6 shows that duration decreases with the coupon, so b) is correct. As the yield increases, the weight of the payments further into the future decreases, which decreases (not increases) the duration. So, c) is false.

Example 1.14: FRM Exam 2004—Question 16

c. Higher duration is associated with physical characteristics that push payments into the future (i.e., longer term, lower coupons, and less frequent coupon payments, as well as lower yields, which increase the relative weight of payments in the future).

Example 1.15: FRM Exam 2001—Question 104

b. With a fixed coupon, the duration goes up to the level of a with the same coupon. See Figure 1.7.

Example 1.16: FRM Exam 2000—Question 106

a. The nine-year bond (number 5) has shorter duration because the maturity is shortest, at nine years, among comparable bonds. Next, we have to decide between bonds 1 and 2, which only differ in the payment frequency. The semiannual bond (number 2) has a first payment in six months and has shorter duration than the annual bond. Next, we have to decide between bonds 1 and 4, which only differ in the yield. With lower yield, the cash flows further in the future have a higher weight, so that bond 4 has greater duration. Finally, the zero-coupon bond has the longest duration. So, the order is 5-2-1-4-3.

Example 1.17: FRM Exam 2002—Question 57

a. The portfolio dollar duration is $D^*P = \sum x_i D_i^* P_i = +8 \times 2.5 \times \$90{,}000 - 6 \times 3.0 \times \$110{,}000 + 12 \times 3.3 \times \$120{,}000 = \$4{,}572{,}000$. The change in portfolio value is then $-(D * P)(\Delta y) = -\$4{,}572{,}000 \times 0.0025 = -\$11{,}430$.

Example 1.18: FRM Exam 2000—Question 110

c. Because convexity is proportional to the square of time to payment, the convexity of a bond will be driven by the cash flows far into the future. Answer I is correct because the 10-year zero has only one cash flow, whereas the coupon bond has several others that reduce convexity. Answer II is false because the 6% bond with 10-year duration must have cash flows much further into the future, say in 30 years, which will create greater convexity. Answer III is false because convexity grows with the square of time. Answer IV is false because some bonds, for example MBSs or callable bonds, can have negative convexity. Answer V is correct because convexity must be positive for coupon-paying bonds.

APPENDIX: APPLICATIONS OF INFINITE SERIES

When bonds have fixed coupons, the bond valuation problem often can be interpreted in terms of combinations of infinite series. The most important infinite series result is for a sum of terms that increase at a geometric rate:

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a} \tag{1.35}$$

This can be proved, for instance, by multiplying both sides by $(1 - a)$ and canceling out terms.

Equally important, consider a geometric series with a finite number of terms, say N. We can write this as the difference between two infinite series:

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = (1 + a + a^2 + a^3 + \cdots) - a^N(1 + a + a^2 + a^3 + \cdots) \tag{1.36}$$

such that all terms with order N or higher will cancel each other.

We can then write

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = \frac{1}{1-a} - a^N \frac{1}{1-a} \tag{1.37}$$

These formulas are essential to value bonds. Consider first a consol with an infinite number of coupon payments with a fixed coupon rate c. If the yield is y and the face value F, the value of the bond is

$$\begin{aligned}
P &= cF\left[\frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] \\
&= cF\frac{1}{(1+y)}[1 + a^2 + a^3 + \cdots] \\
&= cF\frac{1}{(1+y)}\left[\frac{1}{1-a}\right] \\
&= cF\frac{1}{(1+y)}\left[\frac{1}{(1-1/(1+y))}\right] \\
&= cF\frac{1}{(1+y)}\left[\frac{(1+y)}{y}\right] \\
&= \frac{c}{y}F
\end{aligned}$$

Similarly, we can value a bond with a *finite* number of coupons over T periods at which time the principal is repaid. This is really a portfolio with three parts:

1. a long position in a consol with coupon rate c
2. a short position in a consol with coupon rate c that starts in T periods
3. a long position in a zero-coupon bond that pays F in T periods

Note that the combination of (1) and (2) ensures that we have a finite number of coupons. Hence, the bond price should be:

$$P = \frac{c}{y}F - \frac{1}{(1+y)^T}\frac{c}{y}F + \frac{1}{(1+y)^T}F = \frac{c}{y}F\left[1 - \frac{1}{(1+y)^T}\right] + \frac{1}{(1+y)^T}F \tag{1.38}$$

where again the formula can be adjusted for different compounding methods.

This is useful for a number of purposes. For instance, when $c = y$, it is immediately obvious that the price must be at par, $P = F$. This formula also can be used to find closed-form solutions for duration and convexity.