Trigonometry

According to the *Mathematics: Content Knowledge (0061) Test at a Glance* (www.ets.org/Media/Tests/PRAXIS/pdf/0061.pdf), the Trigonometry content category of the Mathematics CK tests your knowledge and skills in five topic areas:

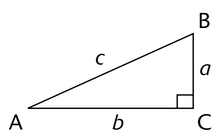
- The six basic trigonometric functions
- The law of sines and the law of cosines
- Special angle formulas and identities
- Trigonometric equations and inequalities
- Rectangular and polar coordinate systems

This review discusses the key ideas and formulas in each topic area that are most important for you to know for the Mathematics CK.

The Six Basic Trigonometric Functions

For this topic, you must be able to define and use the six basic trigonometric functions using the degree or radian measure of angles and know their graphs and be able to identify their periods, amplitudes, phase displacements or shifts, and asymptotes (*Mathematics: Content Knowledge (0061) Test at a Glance*, page 4).

To define the six basic trigonometric ratios, begin with a right triangle ABC and label its parts as follows:



A = measure of $\angle A$
B = measure of $\angle B = 90° - A$
C = measure of $\angle C = 90°$
a = side opposite $\angle A$
b = side adjacent to $\angle A$
c = side opposite the right angle = hypotenuse

The basic formulas relative to angle A in right triangle ABC are as follows:

sine of $\angle A = \sin A = \dfrac{\text{side opposite}}{\text{hypotenuse}} = \dfrac{a}{c}$ cosecant of $\angle A = \csc A = \dfrac{\text{hypotenuse}}{\text{side opposite}} = \dfrac{c}{a}$

cosine of $\angle A = \cos A = \dfrac{\text{side adjacent}}{\text{hypotenuse}} = \dfrac{b}{c}$ secant of $\angle A = \sec A = \dfrac{\text{hypotenuse}}{\text{side adjacent}} = \dfrac{c}{b}$

tangent of $\angle A = \tan A = \dfrac{\text{side opposite}}{\text{side adjacent}} = \dfrac{a}{b}$ cotangent of $\angle A = \cot A = \dfrac{\text{side adjacent}}{\text{side opposite}} = \dfrac{b}{a}$

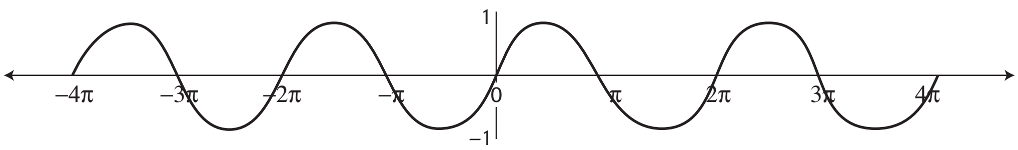
From the preceding formulas, you can see that sine and cosecant are reciprocals of each other, that cosine and secant are reciprocals of each other, and that tangent and cotangent are reciprocals of each other. Therefore, it is necessary to remember only the sine, cosine, and tangent ratios because the other ratios can be determined by using the reciprocal relationships.

Tip: The mnemonic SOH-CAH-TOA (pronounced "soh-kuh-toh-uh") can help you remember that S (sine) is O (opposite) over H (hypotenuse), that C (cosine) is A (adjacent) over H (hypotenuse), and T (tangent) is O (opposite) over A (adjacent).

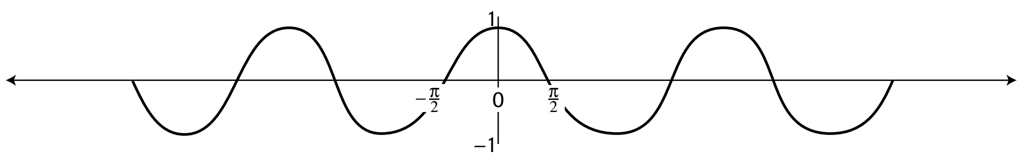
Typically, when working with right triangles on the Mathematics CK, you need to find a missing angle or a missing side of a right triangle. If you are given two sides of the right triangle, you should use the Pythagorean theorem to find

angle (°)	*angle (radians)*	*sine*	*cosine*	*tangent*	*cotangent*	*secant*	*cosecant*
0	0	0	1	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1

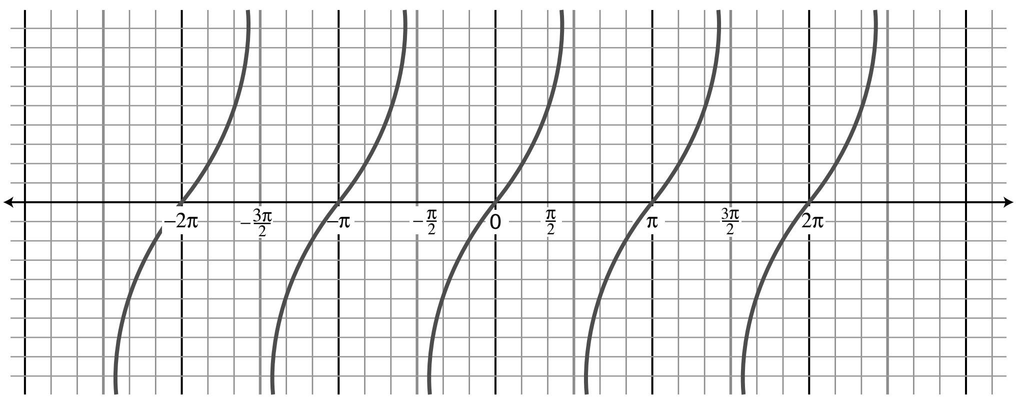
The graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$, the three main trigonometric functions, are shown in the following figures.



$y = \sin x$



$y = \cos x$



$y = \tan x$

The general forms for the sine and cosine functions, $y = a\sin(bx - c) + k$ and $y = a\cos(bx - c) + k$, have graphs with **amplitude** = $|a|$, **period** = $\frac{2\pi}{b}$, a horizontal or **phase shift** of $|\frac{c}{b}|$ units (to the right of the origin if $\frac{c}{b}$ is positive; to the left of the origin if $\frac{c}{b}$ is negative), and a **vertical shift** of $|k|$ units (up from the origin if k is positive; down from the origin if k is negative). Graphically, the coefficient a causes the function to be "stretched" or "shrunk" in the vertical direction by the multiple $|a|$. For the sine and cosine function the maximum height of the graph is $|a|$, and the minimum height is $-|a|$.

The general form for the tangent function, $y = a\tan(bx - c) + k$, has a graph with period = $\frac{\pi}{b}$, a horizontal or phase shift of $|\frac{c}{b}|$ units (to the right of the origin if $\frac{c}{b}$ is positive; to the left of the origin if $\frac{c}{b}$ is negative), and a vertical shift of $|k|$ units (up from the origin if k is positive; down from the origin if k is negative). Graphically, the coefficient a causes the function to be "stretched" or "shrunk" in the vertical direction by the multiple $|a|$; however, unlike the sine and cosine functions, the tangent function has neither a maximum nor a minimum value.

Situation in the Problem	*Use*	*No. of solutions*
You are given the measure of two sides and a non-included acute angle (SSA), and the length of the altitude from the vertex where the two given sides meet falls between the lengths of the two given sides.	Neither	no solution
You are given the measure of three angles (AAA).	Neither	no unique solution

Tip: When solving an oblique triangle, be sure to set the mode of your calculator to degree mode if the angle measurements are in degrees and to radian mode if the angle measurements are given in radians.

Special Angle Formulas and Identities

For this topic, you must be able to apply the formulas for the trigonometric functions of $\frac{x}{2}$, $2x$, $x + y$, and $x - y$ and prove trigonometric identities (*Mathematics: Content Knowledge (0061) Test at a Glance*, page 4).

For the Mathematics CK you need to know certain fundamental identities and formulas that can be used to simplify or change trigonometric expressions. Following is a list of the most important identities to commit to memory before the test.

Reciprocal Identities: $\sec\theta = \frac{1}{\cos\theta}$, $\csc\theta = \frac{1}{\sin\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$

Ratio Identities: $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean Identities: $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, and $\cot^2\theta + 1 = \csc^2\theta$

Cofunction Identities: $\cos\theta = \sin(90° - \theta)$, $\csc\theta = \sec(90° - \theta)$, and $\cot\theta = \tan(90° - \theta)$

The following formulas are given on the formula sheet that will be provided:

Addition Formulas: $\sin(x \pm y) = \sin x\cos y \pm \cos x \sin y$, $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$, and $\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Half-Angle Formulas: $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$ and $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$ (sign depends on the quadrant of $\frac{\theta}{2}$)

Even though you can derive the following formulas from the addition formulas, it is a good idea to be familiar with them.

Double Angle Formulas: $\sin(2\theta) = 2\sin\theta\cos\theta$, $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Since the Mathematics CK is a multiple-choice test, you will not have to prove identities *per se* on the test; however, you likely will have to select which trigonometric expression is an identity for a given trigonometric expression. Here are some strategies you might find helpful.

- Change all the trigonometric functions in the given expression to sines and cosines and simplify.
- Combine fractions and simplify.
- If the numerator or denominator of a fraction has the form $f(x) + 1$, multiply the numerator and denominator by $f(x) - 1$ to obtain the difference of two squares, and then look for a Pythagorean identity; for $f(x) - 1$ multiply by $f(x) + 1$.
- Evaluate the given trigonometric expression for a convenient value of the angle, and then evaluate each of the answer choices for the same value of the angle to find one that evaluates to be the same value as you obtained for the given trigonometric expression.

Trigonometric Equations and Inequalities

For this topic, you must be able to solve trigonometric equations and inequalities (*Mathematics: Content Knowledge (0061) Test at a Glance*, page 4).

Trigonometric equations and inequalities are solved in a manner similar to the way that algebraic equations and inequalities are solved. (See the chapter "Algebra and Number Theory" for a discussion on solving algebraic equations.) The main difference is that due to the periodic nature of the trigonometric functions, there might be multiple solutions to the equation, depending on the specifications given in the problem. For instance, to solve $\cos\theta = \frac{1}{2}$, you must find all values for θ that make the equation true. For simplicity, suppose that you want to express the answer in radians. Since this is the cosine of a special angle (see "The Six Basic Trigonometric Functions" earlier in this chapter for a summary of special angles), you know that θ equal to $\frac{\pi}{3}$ is a solution. However, the cosine function is also positive in QIV. Therefore, the angle $\frac{5\pi}{3}$, which is the angle in QIV with reference angle $\frac{\pi}{3}$ is also a solution. Since you know that the cosine function is periodic with period 2π, you need to add multiples of 2π to each of the values to obtain the full solution set: $\theta = \frac{\pi}{3} + 2\pi k$ and $\theta = \frac{5\pi}{3} + 2\pi k$, where k is any integer. If the problem specifies that you are to find only values in a particular interval, then you omit any values outside the given interval.

You can use your graphing calculator to solve trigonometric equations by using the keys for the inverse trigonometric functions, $y = \sin^{-1}x$, $y = \cos^{-1}x$, and $y = \tan^{-1}x$. The range for an inverse trigonometric function is restricted, so that the inverse will be a function. (See the chapter "Functions" for a discussion on inverses of functions.) The following applies:

Range of Inverse Trigonometric Functions: $\sin^{-1}x$ $[-\pi/2, \pi/2]$; $\cos^{-1}x$ $[0, \pi]$; $\tan^{-1}x$ $(-\pi/2, \pi/2)$

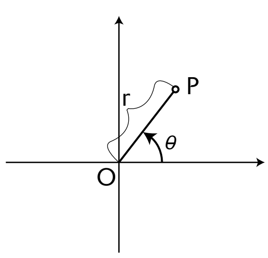
When you use the keys for the inverse functions, the values returned will be in the restricted ranges. You will have to use your knowledge of reference angles and the periodicity of the functions to determine other values in the solution set if needed.

Tip: If the answer choices are given as radians expressed in terms of π, you might set your calculator to degree mode when obtaining the value of the angle, then convert the angle into radians using the relationship that $1° = \frac{p}{180}$ radians. If you set your calculator to radian mode, the value of the solution will be displayed as a real number. You will have to convert the answer choices to real numbers to decide which answer is the same as your solution.

Rectangular and Polar Coordinate Systems

For this topic, you must be able to convert between rectangular and polar coordinate systems (*Mathematics: Content Knowledge (0061) Test at a Glance*, page 4).

In a **polar coordinate system,** a point ***P*** is represented by an ordered pair of coordinates **(r, θ).** The number r is the distance of the point from the origin at point **O,** and θ is the smallest nonnegative angle (measured in a counterclockwise direction) that **OP** makes with the positive horizontal axis.



To change from rectangular (x, y) to polar (r, θ): $r^2 = x^2 + y^2$; $\tan\theta = \frac{y}{x}$ if $x \neq 0$.