(a) If $a_{ij} \geq 0$ for each $i, j \in \mathbb{N}$, then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \sum_{(i,j)\in\mathbb{N}^2} a_{ij},$$

where the last expression above is a standard unordered sum from advanced calculus. [This is a result that is demonstrated in advanced calculus courses independently of any measure or integration theory, and hence our usage of it before this exercise is not in any way circular. Here, the idea is to prove this by constructing the appropriate space and functions and using the results of the previous exercise.]
(b) Let $\{a_{nm} : n, m \in \mathbb{N}\}$ denote a double sequence of nonnegative numbers. Show that $\sum_{n=1}^{\infty} \underline{\lim}_m\, a_{nm} \leq \underline{\lim}_m \sum_{n=1}^{\infty} a_{nm}$.
Comment: For (a), let $\Omega = \{\omega_{ij} : i, j \in \mathbb{N}\}$, $\mathcal{F} = 2^{\Omega}$, and define $f : \Omega \to [0, +\infty]$ by writing $f(\omega_{ij}) = a_{ij}$ for each $i, j \in \mathbb{N}$. Then $f \in \mathfrak{N}$ and

$$\bar{\mathcal{I}}(f) = \sum_{\omega_{ij}\in\Omega} f(\omega_{ij}) = \sum_{i,j\in\mathbb{N}} a_{ij}.$$

Let $A_i = \{a_{i1}, a_{i2}, \cdots\}$ for each $i \in \mathbb{N}$, so that $A_1, A_2, \cdots \in \mathcal{F}$ are disjoint with $\bigcup_{i=1}^{\infty} A_i = \Omega$. Observe that

$$\bar{\mathcal{I}}(f) = \bar{\mathcal{I}}\left(\sum_{i=1}^{\infty} f I_{A_i}\right) = \sum_{i=1}^{\infty} \bar{\mathcal{I}}_{A_i}(f) = \sum_{i=1}^{\infty} \sum_{\omega_{ij}\in A_i} f(\omega_{ij}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}.$$

Similar reasoning gives $\bar{\mathcal{I}}(f) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

Comment for the next three exercises: The following three exercises are important in that they explore equivalent definitions of the operator $\bar{\mathcal{I}}$ as given in this section. Different books on analysis and the Lebesgue integral define the Lebesgue integral differently, and it is useful to see that these alternative definitions are in fact equivalent definitions.

9. Consider the measure space $(\Omega, \mathcal{F}, \mu)$, and let $E \in \mathcal{F}$. For each $f \in \mathcal{N}$, define

$$\bar{\mathcal{J}}_E(f) = \sup \sum_{i=1}^{n} \inf\{f(\omega) : \omega \in A_i \cap E\}\mu(A_i \cap E),$$

where the supremum is taken over all finite disjoint collections $\{A_1, \cdots, A_n\}$ of $\mathcal{F}$-sets with $\bigcup_{i=1}^{n} A_i = \Omega$. Then $\bar{\mathcal{J}}_E(f) = \bar{\mathcal{I}}_E(f)$, so that this is an equivalent formulation of what we have done in this section. [Recall the arithmetical conventions $+\infty \times 0 = 0 \times +\infty = 0$ and $\inf \varnothing = +\infty$ for the case when some $A_i = \varnothing$.]

10. Consider the measure space $(\Omega, \mathcal{F}, \mu)$, and let $E \in \mathcal{F}$ and $f \in \mathfrak{N}$. Define $\bar{\mathcal{J}}_E(f) = \lim_n \mathcal{I}_E(s_n)$, where $\{s_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of functions in $\mathfrak{S}_f$ such that $s_n \to f$. We will show that $\bar{\mathcal{J}}_E(f) = \bar{\mathcal{I}}_E(f)$ *without using MCT*, so that this definition could have instead served as our definition.
(a) We first want to show that $\bar{\mathcal{J}}_E(f)$ is well-defined, since there are possibly many nondecreasing sequences $\{s_n\}_{n=1}^{\infty}$ of elements in $\mathfrak{S}_f$ with $s_n \to f$. Let $t \in \mathfrak{S}_f$, so that $t \leq \lim_n s_n = f$. If $\mathcal{I}_E(t) = +\infty$, then $\lim_n \mathcal{I}_E(s_n) = +\infty$.