regarding forces, impulses, or momenta. The center of mass of a system of objects, whether in two- or three-dimensional space is given as the following:

$$\mathbf{d_{CM} = \Sigma\, m\, r\, /\, M}$$

where $\mathbf{d_{CM}}$ is the distance of the center of mass from the origin or reference point; **m** is the mass of the individual particle; **r** is its distance from the origin or reference point; and **M** is the total mass of all of the particles in the system. If you are finding the center of mass of particles in a two-dimensional system, you find $\mathbf{d_{CM}}$ for ***x*** and then $\mathbf{d_{CM}}$ for ***y***. Using the Pythagorean Theorem, you will obtain the magnitude of the vector from the origin or reference point to the center of mass of the system. For a two-dimensional system, the angle made with the *x*-axis by the center of mass of a system of particles is found by letting the ratio of the y-component to the x-component equal the tangent of the angle. This will yield the vector location of the angle that the center of mass makes with the positive ***x***-axis. For three-dimensional systems, add $\mathbf{d_{CM}}$ for **z** and use the triple Pythagorean to find the magnitude of the resultant vector. To find the angle that the resultant vector makes with the ***x***-axis, for example, use the dot product method discussed in the first chapter.

Example (C Exam)

Find the center of mass for the following three particles: A: 5 kg at (1, -2, 4) cm; B: 4 kg at (–2, 0, 5) cm; C: 3 kg at (–3, 2, 6) cm.

Solution

Particle	*Mass (kg)*	*x (cm)*	*y (cm)*	*z (cm)*
A	5	1	–2	4
B	4	–2	0	5
C	3	–3	2	6
Σ:	12	–4	0	15

$$\mathbf{d_{CM}x} = \frac{\Sigma m\mathbf{x}}{\Sigma m} = \frac{(5\,\text{kg})(1\,\text{cm}) + (4\,\text{kg})(-2\,\text{cm}) + (3\text{kg})(-3\text{cm})}{12\,\text{kg}} = -1\text{ cm }\mathbf{x}$$

$$\mathbf{d_{CM}y} = \frac{\Sigma m\mathbf{y}}{\Sigma m} = \frac{(5\,\text{kg})(-2\,\text{cm}) + (4\,\text{kg})(0\,\text{cm}) + (3\,\text{kg})(2\,\text{cm})}{12\,\text{kg}} = -0.33\text{ cm }\mathbf{y}$$

$$\mathbf{d_{CM}z} = \frac{\Sigma m\mathbf{z}}{\Sigma m} = \frac{(5\,\text{kg})(4\,\text{cm}) + (4\,\text{kg})(5\,\text{cm}) + (3\,\text{kg})(6\,\text{cm})}{12\,\text{kg}} = 4.8\text{ cm }\mathbf{z}$$

The center of mass of the three particles is located at (–1, –0.33, 4.8) cm.

Magnitude of the center of mass vector: $[(-1)^2 + (-0.33)^2 + (4.8)^2]^{\frac{1}{2}}$ = **4.9 cm**

Example (B & C Exams)

Object A has a mass of 5.0 kg and is moving horizontally with velocity 5.0 m/s. Object A is hit by another object, which causes it to reverse direction. If its new velocity is –10 m/s and the objects were in contact with each other for 0.02 seconds, find the average force, **F,** exerted on object A.

Solution

Take the initial direction of object A to be positive.

Impulse produces momentum as shown in the following equation.

$$\mathbf{F}\Delta t = m\Delta\mathbf{v}$$

$$0.02\text{ sec} = (5.0\text{ kg})(\mathbf{v}_F - \mathbf{v}_o)$$

$$(-10\text{ m/s} - 5.0\text{ m/s})$$

$$\mathbf{F} = \frac{m\Delta\mathbf{v}}{\Delta t} = \frac{(5.0\text{ kg})(-15.0\text{ m/s})}{0.02\text{ sec}} = \mathbf{-3750\ N}$$

(The negative sign indicates that the force is applied opposite the initial momentum of object A.)

Angular Momentum

As in linear momentum, **angular** momentum is the product of an object's effective mass and velocity. In this case, however, the effective mass is a result of the *distribution* of mass as an object spins. Also, the velocity is now the **angular** velocity of the object. A rotating object's angular momentum will remain constant unless acted upon by an outside torque. The relationship between **linear** momentum and **angular** momentum, **L,** is described mathematically as follows:

Linear Impulse	Linear Momentum	Angular Impulse	Angular Momentum
$\mathbf{F}\Delta t$	$\mathbf{p} = m\mathbf{v}$	$\tau\Delta t$	$\mathbf{L} = I\boldsymbol{\omega}$

Common representations of **rotational inertia, I,** are

- Point mass, hoop, thin ring, hollow cylinder $I = m\mathbf{r}^2$ *
- Solid disk or solid cylinder(flywheel) $I = \left(\frac{1}{2}\right) m\mathbf{r}^2$ *
- Solid sphere $I = \left(\frac{2}{5}\right) m\mathbf{r}^2$ *
- Hollow sphere $I = \left(\frac{2}{3}\right) mr^2$
- Thin rod of mass, m, and length, l, through center $I = \left(\frac{1}{12}\right) ml^2$
- Thin rod of mass, m, and length, l, at end $I = \left(\frac{1}{3}\right) ml^2$
- Thick ring or washer, with central hole of radius $\mathbf{r}_1$ and washer radius $\mathbf{r}_2$ $I = \left(\frac{1}{2}\right) m(\mathbf{r}_1^2 + \mathbf{r}_2^2)$
- Solid disk with axis through diameter $I = \left(\frac{1}{4}\right) m\mathbf{r}^2$

*The most commonly used rotational inertias

Example (B & C Exams)

A child's toy merry-go-round has a mass of 32 kg, and its plastic platform has a diameter of exactly 1.0 m. What constant unbalanced force, **F**, is required to increase its speed from 0.5 rev/sec to 1.5 rev/sec in 3.0 sec?

Solution

$$\boldsymbol{\tau} = \mathbf{I}\boldsymbol{\alpha} = \mathbf{Fr}$$

$$\left(\frac{1}{2}\right) m\mathbf{r}^2 \frac{(\omega_F - \omega_o)}{\Delta t} \quad \frac{d}{2}$$

$$\mathbf{F} = \frac{\left[\left(\frac{1}{2}\right) m\mathbf{r}^2\right]\left[\left(\frac{\omega_F - \omega_O}{\Delta t}\right)\right]}{0.5\text{ m/rad}} = \frac{\dfrac{\left[\left(\frac{1}{2}\right)(32\text{ kg})(0.5\text{ m})^2\right]\left[(1.5\text{ rev/s} - 0.5\text{ rev/s})(2\pi\text{ rad/rev})\right]}{(3.0\text{ sec})}}{(0.5\text{m/rad})} = \mathbf{16.8\ N}$$

Example (B & C Exams)

A force of 10 N is applied to a 5-kg crate, pushing it a distance of 5 m at a constant speed. How much work is done by the force?

Solution

W = **Fd** since the force is in the direction of motion. The work done is **W** = (10 N)(5 m) = **50 J**

Example (B & C Exams)

If the applied force in the preceding problem is 10 N and the coefficient of kinetic friction between the crate and the surface is 0.2, what is the work done by the friction?

Solution

The work done is the frictional force times the distance. Since $\mu_K = \mathbf{F_F} / \mathbf{F_N}$, $\mathbf{W} = (\mu_K)(\mathbf{F_N})(5\text{ m}) = (0.2)(5\text{ kg})(9.8\text{ m/s}^2)(5\text{ m}) = \mathbf{-49 J}$. (The negative sign indicates work done by friction opposes motion.)

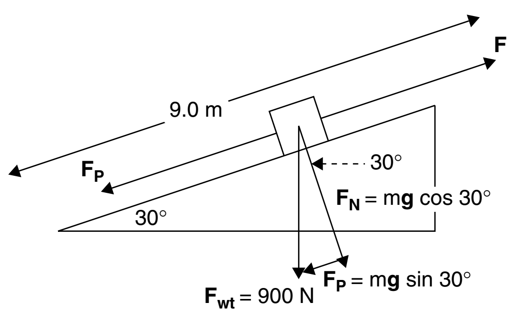
If a force is applied to an object and it does not move in the direction of the applied force, **no work is done by that force.**

Example (B & C Exams)

How much work is done by a planet keeping a satellite in circular orbit around it?

Solution

Since the force of gravitation is toward the planet and the motion of the satellite is perpendicular to that force, it does *no work*.



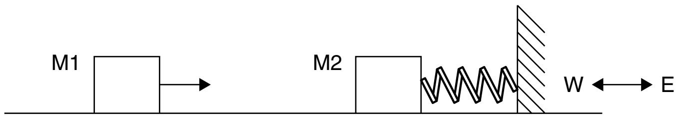
Example (B & C Exams)

A box weighing 900 N is pushed up a ramp a distance of 9.0 m at a constant speed. The ramp is inclined at an angle of 30° with the horizontal, and the coefficient of kinetic friction between the box and the ramp is 0.23. Calculate the work that is done in pushing the box.

Solution

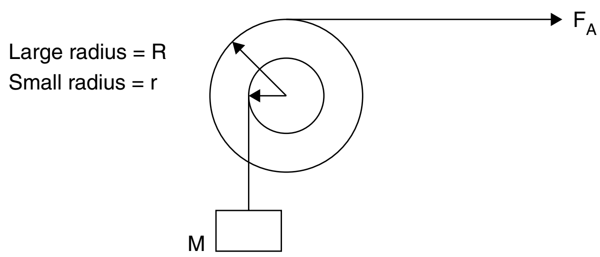
The work done is **Fd**. The force **F** is equal to the sum of the frictional force $\mathbf{F}_F$ and the parallel force $\mathbf{F}_P$. The distance is 9.0 m. This yields the following:

$$W = (\mathbf{F}_F + \mathbf{F}_P)(\mathbf{d}) = [(\mu_K \mathbf{F}_N) + (m\mathbf{g}\sin 30°)](\mathbf{d})$$
$$=[(0.23)(900\text{N})(\cos 30°) + (900\text{ N})(\sin 30°)](9.0\text{ m})$$
$$= [179\text{ N} + 450\text{ N})(9.0\text{ m}) = \mathbf{5700\ J}$$



3. Two equal masses of 20 kg each, M 1 and M 2, sit on a frictionless air track. M 1 is moving east at 0.8 m/s and collides with stationary M 2, which is connected to a wall by a massless spring with k = 2.5 N/m.

(a) If the masses collide elastically, what is the total energy of the system immediately after impact?
(b) Describe the motion after M 1 collides elastically with M 2.
(c) How far does M 2 compress the spring before stopping?
Now the masses are reset and adhesive is applied. M 1 sticks to M 2 upon collision.
(d) What is the combined velocity of the masses at the moment the spring begins to compress?
(e) What is the momentum of the system immediately after impact?
(f) How much work does the spring do in reversing the direction of the combined masses?



4. Two frictionless pulleys are attached rigidly at and are free to rotate about their central axes. The larger pulley has a radius, **R,** and the smaller one has a radius, **r**. A mass, M, hangs from the smaller pulley, and a force, $\mathbf{F_A}$, is applied at a right angle to the top of the larger pulley. Describe all values in terms of M, **g**, **F**, I, **R** and **r**. (I is the system's rotational inertia.)

(a) What force, $\mathbf{F_A}$**,** is necessary to lift M at constant speed?
(b) If $\mathbf{F_A}$ disappears, what is the angular acceleration of the system?
(c) If $\mathbf{F_A}$ from part **(a)** is replaced by a new force of $\mathbf{F_A}$ **+ f**, find the angular acceleration of the system.
(d) A frictional force, $\mathbf{F_F}$ is added to the system. Repeat part (**c**) for this case.

Answers to Free-Response Questions

1. **(a)** $m\mathbf{gh} = (0.8\text{ kg})(9.8\text{ m/s}^2)(2.1\text{ m}) = \mathbf{16\ J}$

(b) $m\mathbf{gh} = \left(\frac{1}{2}\right)m\mathbf{v}^2$ Cancel the masses: $\mathbf{v} = \sqrt{2\mathbf{gh}} = [(2)(9.8\text{ m/s}^2)(2.1\text{ m})]^{\frac{1}{2}} = \mathbf{6.4\ m/\ s}$

(c) $E_K = \left(\frac{1}{2}\right)(m)(\mathbf{v}^2) = \left(\frac{1}{2}\right)(0.8\text{ kg})(6.4\text{ m/s})^2 = \mathbf{16\ J}$. Note that $E_K = m\mathbf{gh}$ from part (**a**).

(d) $\mathbf{p} = m\mathbf{v} = (0.8\text{ kg})(6.4\text{ m/s}) = \mathbf{5.1\ kgm/\ s}$

(e) $\mathbf{F}\Delta t = m\Delta\mathbf{v}$ Solve for force: $\mathbf{F} = m\,\frac{\Delta\mathbf{v}}{\Delta t} = \frac{(0.8\text{ kg})(6.4\text{ m/s})}{0.09\text{ sec}} = \mathbf{57\ N}$

(f) $\mathbf{F}\Delta t = m\Delta\mathbf{v}$ Solve for velocity: $\Delta\mathbf{v} = \mathbf{F}\Delta t/\ m = \frac{(57\text{N})(0.09\text{ sec})}{0.8\text{ kg}} = 7.5\text{ m/s}$

$m\mathbf{gh} = \left(\frac{1}{2}\right)m\mathbf{v}^2$ Solve for $\mathbf{h}$: $\mathbf{h} = \mathbf{v}^2/2\mathbf{g} = \frac{(7.5\text{ m/s})^2}{(2)(9.8\text{ m/s}^2)} = \mathbf{2.9\ m}$

2. **(a)** When the cart stops, all of its kinetic energy has become potential energy.

$\left(\frac{1}{2}\right)m\mathbf{v}^2 = m\mathbf{gh}$ and $\mathbf{h} = \mathbf{v}^2 / 2\mathbf{g} = (2.5\text{ m/s})^2/(2)(9.8\text{ m/s}^2) = 0.32$ m high

Since sin 30° = 0.5 = **h** / **d**, this yields **d** = 0.32 m/0.5 = **0.64 m**

(b) $E_P = m\mathbf{gh}/2 = (3.0\text{ kg})(9.8\text{ m/s}^2)(0.32\text{m})/2 =$ **4.7 J**

(c) The cart's kinetic energy is equal to its potential energy halfway up the ramp. To check: $E_K = (1/2)m\mathbf{v}^2/2 = (3.0\text{ kg})(2.5\text{ m/s})^2/4 =$ **4.7 J ANS**

(d) The cart's weight extends the spring to a maximum distance **x**. The force exerted on the cart is –k**x**. This gives m**g** = –k**x** **x** = m**g**/k = $(3.0\text{ kg})(9.8\text{ m/s})^2/90$ N/m = **0.3 m**

3. **(a)** The total energy of the system after the collision is the same as that before the collision. Since M 1 possesses all the system's energy before colliding, the total energy of the system remains equal to $E_{K\,1} = (1/2)(M\,1)(\mathbf{v_O}^2) = (1/2)(20\text{ kg})(0.8\text{ m/s})^2 =$ **6.4 J**

(b) M 1 hits M 2, stops in place, takes the place of M 2, which continues into the spring at 0.8 m/s east, is slowed down and momentarily stopped by the spring, reverses direction, hits M 1 with velocity 0.8 m/s west, stops in place, takes the place of M 1, which continues moving west at 0.8 m/s.

(c) The kinetic energy of the system before impact equals the potential energy ultimately stored in the spring.

E_K of Mass 1 = $\left(\frac{1}{2}\right) k\mathbf{x}^2$.

$$\mathbf{x} = [(2E_K)/k]^{\frac{1}{2}}$$

From part (**a**)'s answer, $\mathbf{x} = [(2)(6.4\text{ J})/(2.5\text{ N/m})]^{1/2} =$ **2.3 m**

(d) Total momentum BEFORE collision equals total momentum AFTER: (20 kg)(0.8 m/s east) = (40 kg)($\mathbf{v_{FINAL}}$) yields $\mathbf{v_{FINAL}}$ **= 0.4 m/ s east**

(e) **p** = m**v** = (40 kg)(0.4 m/s east) = **16 kgm/ s east**

(f) All work done compressing the spring is equal and opposite the work done decompressing it. The sum is zero.

4. **(a)** Since the system is not accelerating, it is in equilibrium and the sum of the clockwise torque and counterclockwise torque equals zero. This means that $\mathbf{F_A R} = M\mathbf{g}r$ and $\mathbf{F_A = Mgr/R}$

(b) Using the general relationship, $\boldsymbol{\tau} = \mathbf{Fr} = \mathbf{I\alpha}$

$$(M\mathbf{g} - M\mathbf{a})r = I\alpha$$
$$m\mathbf{g}r - m\alpha r^2 = I\alpha$$
$$I\alpha + M\alpha r^2 = M\mathbf{g}r$$
$$\alpha(I + Mr^2) = M\mathbf{g}r$$
$$\alpha = \frac{M\mathbf{g}r}{(I + Mr^2)}$$

(c) Again, using the general relationship, $\tau = Fr = I\alpha$

$$(F_A + f)R + M\mathbf{g}r = I\alpha$$
$$\alpha = \frac{(\mathbf{F_A} + \mathbf{f})R + M\mathbf{g}r}{I}$$

(d) The general relationship is used again:

$$\boldsymbol{\tau} = \mathbf{Fr} = \mathbf{I\alpha}$$
$$(\mathbf{F_A} + \mathbf{f} + \mathbf{F_f})R + M\mathbf{g}r = I\alpha$$
$$\alpha = \frac{(\mathbf{F_A} + \mathbf{f} + \mathbf{F_f})R + M\mathbf{g}r}{I}$$