Calculus

According to the *Mathematics: Content Knowledge Test (0061) at a Glance* (www.ets.org/Media/Tests/PRAXIS/pdf/0061.pdf), the Calculus content category of the Mathematics CK tests your knowledge and skills in nine topic areas:

- Limits
- Derivatives
- Continuity
- Analyzing the behavior of a function
- The Mean Value Theorem and the Fundamental Theorem of Calculus
- Integration as a limiting sum
- Differentiation and integration techniques
- Numerical approximation of derivatives and integrals
- Limits of sequences and series

This review will discuss the key ideas and formulas in each topic area that are most important for you to know for the Mathematics CK.

Limits

For this topic, you must demonstrate an understanding of what it means for a function to have a limit at a point; calculate limits of functions or determine that the limit does not exist; and solve problems using the properties of limits (*Mathematics: Content Knowledge (0061) Test at a Glance*, page 4).

The study of calculus begins with the study of limits. For the Mathematics CK, the following definitions and properties of limits are essential to know.

Let f be a function defined at every number in an open interval containing a, except possibly at a, then the $\lim_{x \to a} f(x) = L$ (read "the **limit** of f as x approaches a equals L") if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$. The limit is said to *exist* only if the following conditions are satisfied: (i) the limit L is a single finite real number; and (ii) the limit as x approaches a from the left equals the limit as x approaches a from the right; that is, $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$. If no such L exists, then we say that $\lim_{x \to a} f(x)$ *does not exist.*

Common situations that occur when the limit of a function f as x approaches a does *not* exist are (i) $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, (ii) $f(x)$ increases or decreases without bound as x approaches a, or (iii) $f(x)$ oscillates between two fixed values as x approaches a.

Informally, the "ε-δ" definition of limit given here means that if the values of $f(x)$ get arbitrarily close to a single value L as x approaches a from either side, then $\lim_{x \to a} f(x) = L$. When $\lim_{x \to a} f(x)$ exists, the limit is unique. Furthermore, its value is independent of the value of f at a. When $\lim_{x \to a} f(x)$ exists, three situations might occur at a: (i) $f(a) = \lim_{x \to a} f(x)$; (ii) $f(a)$ is undefined; or (iii) $f(a)$ is defined, but $f(a) \neq \lim_{x \to a} f(x)$.

When $\lim_{x \to a} f(x)$ exists and $\lim_{x \to a} f(x) = f(a)$, you can find the limit by direct substitution. Here are some common limits that can be evaluated using direct substitution.

$\lim_{x \to a} b = b$ $\qquad$ $\lim_{x \to a} x^n = a^n$, for n a positive integer.

$\lim_{x \to a} x = a$ $\qquad$ $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$, for n a positive integer with the restriction that if n is even, $a > 0$.

Derivatives

For this topic, you must demonstrate an understanding of the derivative of a function as a limit, as the slope of a curve, and as a rate of change (for example, velocity, acceleration, growth, decay) (*Mathematics: Content Knowledge (0061) Test at a Glance*, page 4).

The **derivative** $\boldsymbol{f'}$ (read "f prime") of the function f at the number x is defined as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, if this limit exists. If this limit does not exist, then f does not have a derivative at x. A ***differentiable*** function is a function that has a derivative. If $f'(c)$ exists, then f is **differentiable** at c; otherwise, f does not have a derivative at c. Various symbols are used to represent the derivative of a function f. If you use the notation $y = f(x)$, then the derivative of f can be symbolized by $f'(x)$, $D_x f(x)$, y', $D_x y$, $\frac{dy}{dx}$, or $\frac{d}{dx}f(x)$.

The derivative f' is called the **first derivative** of f. The derivative of f' is called the **second derivative** of f and is denoted f''. Similarly, the derivative of f'' is called the **third derivative** of f and is denoted f''', and so on.

If $f'(a)$ exists, then the **tangent line** to the graph of the function f at the point $P(a, f(a))$ is the line through P that has **slope** $m = f'(a)$.

If $f'(t)$ exists, then the (instantaneous) **rate of change** of f at t is $f'(t)$. For example, if $s(t)$ is the position function of a moving object, then the **velocity** (the instantaneous rate of change) of the object at time t is $s'(t)$. Additionally, the **acceleration** of the object at time t is $s''(t)$.

Continuity

For this topic, you must be able to show that a particular function is continuous and demonstrate an understanding of the relationship between continuity and differentiability (*Mathematics: Content Knowledge (0061) Test at a Glance*, page 4).

The function f is **continuous** at the point a in the domain of f if all three of the following conditions are met: (i) $f(a)$ is defined; (ii) $\lim_{x \to a} f(x)$ exists; and (iii) $\lim_{x \to a} f(x) = f(a)$; otherwise, the function f is **discontinuous** at x. A function f is continuous in an open interval if it is continuous at each point in the interval. If a function is continuous on the entire real line, the function is **everywhere continuous;** that is to say, its graph has no holes, jumps, or gaps in it.

The following types of functions are continuous at every point in their domains.

Continuous Functions	
Constant functions	$f(x) = k$, where k is a constant
Power functions	$f(x) = x^n$, where n is a positive integer
Polynomial functions	$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x^1 + a_0$
Rational functions	$f(x) = \frac{p(x)}{q(x)}$, provided $q(x) \neq 0$
Radical functions	$f(x) = \sqrt[n]{x}$, $x \geq 0$, where n is a positive integer
Trigonometric functions	$f(x) = \sin x$, $f(x) = \cos x$, $f(x) = \tan x$, $f(x) = \cos x$, $f(x) = \cot x$, $f(x) = \csc x$

If g is continuous at c and f is continuous at $g(c)$, then $f \circ g(x) = f\left(g(x)\right)$ is continuous at c.

If a function g is differentiable at c, then f is continuous at c; in other words, *differentiability implies continuity*. Therefore, if f is *not* continuous at c, then f is also *not* differentiable at c. *Caution:* Continuity does *not* imply differentiability. A function can be continuous at a point a even though $f'(x)$ does not exist at a. This circumstance occurs when there is a cusp (a sharp corner) or a vertical tangent line at a.